The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations and Linear Algebra

September 9, 2023

Differential Equations

1. (10 points) Acetaminophen, a drug used to manage fever and pain, is cleared from the body at a rate proportional to the amount present with a rate constant k given in units of hour^{-1}. An adult starts having a headache at $t = 0$, takes a pill at time $t = \tau$ and continues to take one every τ hours after that. This continues indefinitely. Treating each pill taken and digested as an instantaneous event, a differential equation to model the mass of acetaminophen $m(t)$ in the person's body as a function of time is

$$\frac{dm}{dt} = -km + m_0 \sum_{n=1}^{\infty} \delta(t - n\tau)$$

with an initial condition $m(0) = 0$.

(a) Use Laplace Transforms to solve the IVP. Express the solution to the ODE in terms of the Heaviside functions $H(t - n\tau)$ where $n = 0, 1, 2, \ldots$

(b) Sketch the graph of the solution.

(c) Note that after a long time, the mass of acetaminophen in the person's bloodstream just after taking a pill asymptotically approaches a constant value. Calculate the asymptotic value of the mass in the blood immediately after taking a pill; that is, find $\lim_{p\to\infty} m(p\tau^+)$ where the limit runs over positive integers p.

Function	Laplace transform
e^{-at}	$\frac{1}{s+a}$
$H(t-c)f(t-c)$	$e^{-cs}F(s)$
$\delta(t-c)$	e^{-cs}

Other useful information:

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

2. (10 points) Consider the PDE, in non-dimensional form,

$$u_t = Du_{xx} + u$$

with Neumann boundary conditions $u_x(0, t) = u_x(1, t) = 0$ and an initial condition $u(x, 0) = f(x)$.

(a) Find an expression for the solution using an appropriate Fourier series.

(b) Under what condition(s) on $f(x)$ does the solution remain bounded? To what function does the solution converge?

3. (10 points) The equations for the shape of a thin elastic rod constrained to the surface of a cylinder that has minimal bending energy is

$$\frac{d\phi}{ds} = p,$$

$$\frac{dp}{ds} = -2(\cos\phi)^3 \sin\phi,$$

where $\phi(s)$ is the angle the tangent line to the rod makes with the horizontal and s is the arclength position along the rod. Physical conditions imposed on the ends of the rod become boundary conditions.

(a) Rewrite the first order system as a single second order ODE for $\phi(s)$. Multiplying the second order equation by $d\phi/ds$ and integrating, find a function $H(\phi, p)$ whose level curves are solutions to the original system.

(b) Find all steady states of the system (consider only a single period of the phase plane). For each steady state, find the eigenvalues of the linearization about it.

(c) Sketch several solution curves in the phase plane, enough of them so that the behaviour of solutions in all regions of the plane is made clear. Specifically, the classification of the steady states should be easily determined from the sketch. State the classification of each steady state (i.e. stable node, unstable spiral etc.). Are the eigenvalues sufficient to determine these classifications? Why or why not?

(d) For a rod of length L with one end ($s = 0$) held with a fixed angle and the other end left untouched, the boundary conditions are $\phi(0) = \phi_0$ and $p(L) = 0$. Describe in words what a solution to such a BVP looks like in the phase plane.

Linear Algebra

4. (10 points) (a) [4 points] Find a lower triangular matrix L such that $LL^T = A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix}$$

(b) [3 points] Compute $\det(L)$.

(c) [3 points] Find the volume in $\mathbb{R}^4$ of the set $S_A = \{x \in \mathbb{R}^4 : x^T A x \leq 1\}$.
You can use the fact that the volume in $\mathbb{R}^4$ of the set $S_I = \{x \in \mathbb{R}^4 : x^T x \leq 1\}$ is $\frac{1}{2}\pi^2$.

5. (8 points) Let P_n be the $n+1$-dimensional space of polynomials of degree n with real coefficients, and let $\langle \cdot, \cdot \rangle$ be the inner product defined as

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

(a) [4 points] Find an orthogonal basis $\{u_0, u_1, u_2\}$ for P_2 such that $u_j \in P_j$, and $u_j(1) > 0$.
(b) [2 points] Using the basis in part (a), express the operator $F[p] := \int_{-1}^{1} p(x)dx$ acting on P_2 as a 1×3 matrix.
(c) [2 points] Using the basis in part (a), express the derivative operator $D[p] := \frac{d}{dx}p(x)$ as a 3×3 matrix.

6. (12 points) Recall that an orthogonal projection matrix is a matrix P that satisfies

$$P^2 = P, \quad P = P^T.$$

Suppose P is an $n \times n$ projection matrix with $\mathrm{rank}(P) = k$. In the following, I_m denotes the $m \times m$ identity matrix.
(a) [4 points] List all the eigenvalues of P, including multiplicity. Be sure to justify your reasoning.
(b) [4 points] Show that $P = AA^T$ for some $n \times k$ matrix A such that $A^T A = I_k$.
(c) [4 points] Suppose P_1, P_2 are two $n \times n$ projection matrices with rank k. Show there exists an $n \times n$ orthonormal matrix U (i.e. such that $U^T U = I_n$, $UU^T = I_n$) such that $P_2 = UP_1U^T$.