Qualifying Exam Problems: Algebra
(Jan 10, 2015)

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1. (10 points) Suppose G is a finite group, p is the smallest prime dividing $|G|$, and H is a subgroup of G with $[G:H]=p$. Show that H is a normal subgroup of G. (Hint: consider the action of G on the set of right cosets of H.)

Solution: Consider the right action of G on the set of right cosets of H in G. This action defines a homomorphism ϕ from G into symmetric group on p letters. Then

$$\operatorname{Ker}\phi = \{g \in G : \quad gkH = kH \quad \forall k \in G\}.$$

Since if $g \in \operatorname{Ker}\phi$, then since in particular $gH = H$, we see that $\operatorname{Ker}\phi \subset H$. Since $G/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$ is a subgroup of S_p, the symmetric group on p letters, we see that $|G/\operatorname{Ker}\phi| = [G : \operatorname{Ker}\phi]$ must divide $p!$. Then since

$$\frac{|G|}{|\operatorname{Ker}\phi|} = \frac{|G|}{|H|}\cdot\frac{|H|}{|\operatorname{Ker}\phi|} = p\cdot\frac{|H|}{|\operatorname{Ker}\phi|}$$

divides $p!$, the prime factors of $|H|/|\operatorname{Ker}\phi|$ must be smaller than p. However, since p is the smallest prime dividing $|G|$, we deduce that $|H|/|\operatorname{Ker}\phi| = 1$ and hence $\operatorname{Ker}\phi = H$. This implies that

$$gkH = kH \quad \forall g \in H,\, k \in G,$$

or in other words

$$k^{-1}gk \in H \quad \forall g \in H,\, k \in G$$

which proves H is normal.

2. (a) (2 points) Show that $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

Solution: Since clearly $K = \mathbb{Q}(\sqrt{2}+\sqrt{3}) \subset \mathbb{Q}(\sqrt{2},\sqrt{3})$, we need to show the reverse inclusion. $(\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6} \in K$ and so $\sqrt{6} \in K$. Thus $\sqrt{6}(\sqrt{2}+\sqrt{3}) = 2\sqrt{3}+3\sqrt{2} \in K$ and so $(2\sqrt{3}+3\sqrt{2}) - 2(\sqrt{3}+\sqrt{2}) = \sqrt{2} \in K$ which then implies that $\sqrt{3} \in K$ so that $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subset K$.

(b) (3 points) Determine the minimal polynomial for $\sqrt{2}+\sqrt{3}$ over the field $\mathbb{Q}$.

Solution: Let $\alpha = \sqrt{3}+\sqrt{2}$. Then $\alpha^2 = 5+2\sqrt{6}$ and so $(\alpha^2-5)^2 = 24$ or equivalently, α satisfies the equation

$$x^4 - 10x^2 + 1 = 0.$$

We claim this is the minimal polynomial. This follows if we show that the degree of the extension $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ is 4. The degree of the extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is 2 since $\sqrt{2} \notin \mathbb{Q}$ so we just need to show that the degree of $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}(\sqrt{2})$ is not 1, i.e. that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. Suppose that

$$\sqrt{3} = a + b\sqrt{2}$$

for $a, b \in \mathbb{Q}$. Then $a^2 + 2b^2 + 2ab\sqrt{2} = 3 \in \mathbb{Q}$ and so $ab\sqrt{2} \in \mathbb{Q}$. Clearly neither a nor b can be zero and so this implies that $\sqrt{2} \in \mathbb{Q}$, a contradiction.

(c) (3 points) Determine the Galois group and all the intermediate field extensions of the extension $\mathbb{Q}(\sqrt{2}+\sqrt{3})/\mathbb{Q}$.

Solution: The roots of the minimal polynomial

$$x^4 - 10x^2 + 1 = (x^2 - 5)^2 - 24$$

are given by

$$+\sqrt{2} + \sqrt{3}, \quad +\sqrt{2} - \sqrt{3}, \quad -\sqrt{2} + \sqrt{3}, \quad -\sqrt{2} - \sqrt{3},$$

and the Galois group must be an order 4 subgroup of the group of permutations of these roots. The automorphisms of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ given by $(\sqrt{2}, \sqrt{3}) \mapsto (-\sqrt{2}, \sqrt{3})$ and $(\sqrt{2}, \sqrt{3}) \mapsto (\sqrt{2}, -\sqrt{3})$ generate an order four group isomorphic to the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$ permuting the roots and so this must be the Galois group.

By the fundamental theorem of Galois theory, the intermediate fields are in bijective correspondence with proper, non-trivial subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/2$ and so there are three of them, corresponding to the fixed fields of the three non-trivial automorphisms:

$$\begin{aligned} (\sqrt{2}, \sqrt{3}) &\mapsto (-\sqrt{2}, \sqrt{3}) \text{ fixes } \mathbb{Q}(\sqrt{3}) \\ (\sqrt{2}, \sqrt{3}) &\mapsto (\sqrt{2}, -\sqrt{3}) \text{ fixes } \mathbb{Q}(\sqrt{2}) \\ (\sqrt{2}, \sqrt{3}) &\mapsto (-\sqrt{2}, -\sqrt{3}) \text{ fixes } \mathbb{Q}(\sqrt{3}\sqrt{2}) = \mathbb{Q}(\sqrt{6}) \end{aligned}$$

(d) (2 points) Determine the splitting field for the polynomial $x^4 + 4$ over $\mathbb{Q}$. What is the degree of this splitting field over $\mathbb{Q}$?

Solution: The polynomial $x^4 + 4$ factors as

$$(x^2 + 2x + 2)(x^2 - 2x + 2)$$

and so the roots are $-1 \pm i$ and $1 \pm i$. Alternatively we can find the roots by solving $x^4 = -4$ by De Moivre's theorem. One root is $x = \sqrt{2}e^{i\pi/4} = 1 + i$, and the others are obtained by multiplying by the 4th roots of unity $\{1, i, -1, -i\}$. The splitting field is therefore $\mathbb{Q}(i)$ which has degree 2.

3. (10 points) Show that the ring $\mathbb{R}[x, y]/(x^2 + y^2 - 2, xy + 1)$ is isomorphic to the ring $\mathbb{R}[a]/(a^2) \oplus \mathbb{R}[b]/(b^2)$. Hint: draw the curves $\{x^2 + y^2 = 2\} \subset \mathbb{R}^2$ and $\{xy = -1\} \subset \mathbb{R}^2$.

Solution: Geometrically, the two curves are tangent to each other at the points $(1, -1)$ and $(-1, 1)$ suggesting that a rotation by $\pi/4$ will simplify the problem. Thus we let $x = w + z$ and $y = w - z$ inducing an isomorphism

$$\begin{aligned} \mathbb{R}[x, y]/(x^2 + y^2 - 2, xy + 1) &\cong \mathbb{R}[w, z]/(w^2 + z^2 - 1, w^2 - z^2 + 1) \\ &\cong \mathbb{R}[w, z]/(w^2, z^2 - 1). \end{aligned}$$

We then define a ring homomorphism

$$\phi : \mathbb{R}[w, z]/(w^2, z^2 - 1) \to \mathbb{R}[a]/(a^2) \oplus \mathbb{R}[b]/(b^2)$$

by

$$\phi : f(w, z) \mapsto (f(a, 1), f(b, -1)).$$

ϕ is well-defined since $\phi(w^2) = \phi(z^2 - 1) = (0, 0)$. We just need to show that ϕ is surjective and injective. The ring $\mathbb{R}[w, z]/(w^2, z^2 - 1)$, considered as a vector space over $\mathbb{R}$, is spanned by $\{1, w, z, wz\}$ since any monomials divisible by z^2 or w^2 can be reduced in degree using the relations. Similarly, the vector space underlying the ring $\mathbb{R}[a]/(a^2) \oplus \mathbb{R}[b]/(b^2)$ is spanned by $\{(1,0),(0,1),(a,0),(0,b)\}$. As a linear transformation, expressed in the given basis, ϕ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

which has determinant -4 and is hence invertible. Thus ϕ is injective and surjective.

4. a) (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with all eigenvalues greater than or equal to 0. Show that there exists a square matrix B with $A = B^T B$.

   b) (3 points) show that for any square matrix $C \in M_{n\times n}(\mathbb{R})$, the matrix $C^T C$ is a symmetric matrix with all eigenvalues greater than or equal to 0.

   c) (2 points) Find the Jordan Canonical form of the matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$$

Solution: If A is a symmetric matrix then there exists and orthogonal matrix Q and a diagonal matrix D (of eigenvalues $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0, \ldots$) such that $AQ = QD$ or $A = QDQ^T$. Now since

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix} \text{ we may take } E = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\lambda_3} & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

(using the fact that $\lambda_i \geq 0$) so that $EE^T = E^2 = D$. Then $A = QEE^TQ^T = B^TB$ for $B = E^TQ^T$.

Now if C^TC is symmetric since $\left(C^TC\right)^T = C^T\left(C^T\right)^T = C^TC$ and so C^TC has real eigenvalues. Let $(C^TC)\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. But then $\mathbf{x}^T(C^TC)\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x}$ and so $||C\mathbf{x}|| = \mathbf{x}^T(C^TC)\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda||\mathbf{x}||$. Given that $||C\mathbf{x}|| \geq 0$ and $||\mathbf{x}|| > 0$ we deduce that $\lambda \geq 0$.

For c), the characteristic polynomial is $\det(xI - A) = (x-1)^3$. One can then proceed to find the dim of the subspaces. Easy to check that the answer is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5. (10 points) Let

$$A = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{bmatrix}$$

The eigenvalues for A are -1,5. Determine an orthonormal basis for $\mathbb{R}^3$ that are eigenvectors for A and then give an orthogonal matrix Q and a diagonal matrix D so that $AQ = QD$.

Solution: By a variety techniques we can determine $\det(A - \lambda) = -(\lambda + 1)^2(\lambda - 5)$. Either use a straightforward calculation or recall that the determinant is the product of the eigenvalues or that the trace is the sum of the eigenvalues. The issue here is determining two orthogonal vectors in the eigenspace for $\lambda = -1$.

$$\lambda = -1 \ : \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and } \lambda = 5 \ : \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The students might use Gram Schmidt on the 2-dimensional eigenspace or perhaps using the cross product given two eigenvectors. Now the students need to remember to normalize to obtain

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

6. Let $A \in M_{n\times n}(\mathbb{R})$. Define the map $f : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by

$$f(A) = A^T.$$

(a) (2 points) Show that f is linear.

Solution: We check that $f(A + B) = (A + B)^T = A^T + B^T = f(A) + f(B)$ and $f(kA) = (kA)^T = kA^T = kf(A)$.

(b) (3 points) Determine the dimension of the eigenspace of eigenvalue 1 for f.

Solution: If A is an eigenvector of eigenvalue 1 then $f(A) = A$ and so $A^T = A$ and so A is symmetric. The dimension of the space of symmetric matrices is $\binom{n}{2} + n$, namely the matrices $E_{ij} + E_{ji}$ for $i \neq j$ and E_{ii} (where E_{ij} is the matrix in $M_{n\times n}(\mathbb{R})$ with a 1 in position i, j and 0's elsewhere.

(c) (3 points) A matrix C is *skew symmetric* if $C^T = -C$. Determine the dimension of the eigenspace of eigenvalue -1 for f.

Solution: If A is an eigenvector of eigenvalue -1 then $f(A) = -A$ and so $A^T = -A$ and so A is skew symmetric. As above the dimension of the space of skew symmetric matrices is $\binom{n}{2}$, namely the matrices $E_{ij} - E_{ji}$ for $i \neq j$.

(d) (2 points) Show that any matrix $A \in M_{n\times n}(\mathbb{R})$ is a sum of a symmetric matrix B and a skew symmetric matrix C.

Solution: Using arguments about eigenspaces we note that the eigenspaces of different eigenvalues are linearly independent namely the eigenspaces for 1 and -1 generate a vector space of dimension $\binom{n}{2} + n + \binom{n}{2} = n^2$ which is the dimension of $M_{n\times n}(\mathbb{R})$. So a basis for the eigenspace for eigenvalue 1 and a basis for the eigenspace for eigenvalue -1 yield a basis for $M_{n\times n}(\mathbb{R})$ and so every $A \in M_{n\times n}(\mathbb{R})$ can be written as a sum $B + C$ where B is symmetric and C is skew symmetric.
Alternatively one can note that $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.