The University of British Columbia
Department of Mathematics
Qualifying Examination—Linear Algebra and Differential Equations
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1. (a) Find A^{10} for the matrix A given below:

$$A = \begin{bmatrix} -7 & -3 \\ 18 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}.$$

   (b) Let $P_2[x]$ denote the set of all polynomials in x having real coefficients and degree at most 2. Determine a basis for $P_2[x]$ which contains both $1 + x + x^2$ and $2 + x + x^2$.

   (c) Find all possible values of $\det(A + A^{-1})$, allowing arbitrary 3×3 matrices A with eigenvalues -1, 1, 2.

2. Let $M_{3\times3}$ be the vector space consisting of all 3×3 matrices (with elementwise addition and the usual scalar multiplication). Let 0 denote the zero matrix in $M_{3\times3}$. For each matrix A in $M_{3\times3}$, define

$$Z(A) = \{B \in M_{3\times3} \,:\, BA = 0\}.$$

   (a) Show that for each fixed $A \in M_{3\times3}$, the set $Z(A)$ is a subspace of $M_{3\times3}$.

   (b) Find the dimension of $Z(0)$, where 0 denotes the zero matrix in $M_{3\times3}$.

   (c) Suppose $A \in M_{3\times3}$ and $\text{rank}(A) = 2$. Find $\dim(Z(A))$.

3. Let k vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^d$ be given. Assume that, for some constant $\alpha \in (0, 1)$,

$$\mathbf{u}_i^T \mathbf{u}_j = \begin{cases} 1, & \text{if } i = j, \\ \alpha, & \text{if } i \neq j. \end{cases}$$

   (Such a collection of unit vectors is called *equiangular*. Notice that the statement $\mathbf{u}_j^T \mathbf{u}_j = 1$ implicitly specifies that each $\mathbf{u}_j$ is a *column* vector of unit length.)

   Consider the set of $d \times d$ matrices $S = \left\{\mathbf{u}_i \mathbf{u}_i^T \,:\, i = 1, 2, \ldots, k\right\}$. Prove the following.

   (a) If the matrices in S are linearly independent, then $k \leq \dfrac{d(d+1)}{2}$.

   (b) The matrices in S are, in fact, linearly independent.
   Hint: One approach starts by postulating the matrix equation $\sum_{i=1}^{k} a_i \mathbf{u}_i \mathbf{u}_i^T = 0$, then multiplying from the left by $\mathbf{u}_j^T$ and from the right by $\mathbf{u}_j$.

4. Consider the following system of nonlinear ordinary differential equations:

$$\begin{cases} x' = y, \\ y' = x - x^2. \end{cases}$$

   (a) Find and classify all equilibrium/fixed points.

   (b) Find a function $V(x, y)$ that is constant along every system trajectory.

   (c) Find the equation of the orbit (also referred to as a homoclinic orbit) that separates the closed orbits from the open ones in the (x, y)-plane.

   (d) Sketch the global phase portrait in the phase space (i.e., in the (x, y)-plane).

5. Solve the following nonhomogeneous wave equation using standard separation-of-variables methods:

(PDE)	$u_{tt} = u_{xx} + \pi^2 \sin \pi x,$	$0 < x < 1,$	$t > 0,$
(BC)	$u(0,t) = 0,\ u_x(1,t) = -\pi,$		$t > 0,$
(IC)	$u(x,0) = 2 \sin \pi x,$	$0 < x < 1,$	
	$u_t(x,0) = 0,$	$0 < x < 1.$	

6. In a simple biological model, the region Ω defined by $x^2 + y^2 + z^2 < R^2$ represents a cell filled with a homogeneous fluid, and the goal is to model the concentration of a certain chemical in Ω. This concentration, C, depends on both time and location, according to

$$\frac{\partial C}{\partial t} = C_0 - C + \nabla^2 C. \qquad (*)$$

Here the constant C_0 represents the initial concentration: at time $t = 0$, $C = C_0$ at each point in Ω. We consider a radially symmetric situation, in which $C = C(r,t)$ for $r = \sqrt{x^2 + y^2 + z^2}$. On the cell wall, where $r = R$, the chemical enters the cell at a constant rate, leading to the boundary condition

$$\left.\frac{\partial C}{\partial r}\right|_{r=R} = J, \qquad (\dagger)$$

where J is a constant. Recall that, for functions of the radial coordinate only, the Laplacian in spherical coordinates has the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right).$$

(a) Find a change of dependent variable that transforms the given PDE $(*)$ into

$$\frac{\partial U}{\partial t} = -U + \frac{\partial^2 U}{\partial r^2}. \qquad (**)$$

Also find the corresponding boundary conditions (BC's) for U at both $r = 0$ and $r = R$. (*Hint*: Pure inspiration is recommended. Failing that, seek $f = f(r)$ so that $C = C_0 + f(r)U(r,t)$.)

(b) Find a second change of variable that transforms the PDE $(**)$ into a standard diffusion equation, $u_t = u_{rr}$. What are the boundary conditions for u?

(c) In the case where $J = 0$ in $(\dagger)$, the conditions on u in part (b) lead to an eigenvalue problem. State this eigenvalue problem, and show that it has Sturm-Liouville type, but do not try to solve it.

(d) Find the equilibrium (or steady-state) concentration, $C = C(r)$, in terms of the constant J in $(\dagger)$.