The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations

January, 2019

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1. (12 points) Some questions on eigenvalues.
   (a) For the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, determine the eigenvalues, their algebraic multiplicities, and their geometric multiplicities.
   (b) A 2×2 matrix B has eigenvalues $1 + i$ and $1 - i$. Find B^4 and explain your answer.
2. (15 points) Consider a 2×2 matrix A with real entries.
   (a) What are the possible values for $\text{rank}(A)$?
   (b) For each of the values of $\text{rank}(A)$ in (a), make two schematic diagrams (sketches): the first should show the nullspace (or kernel) of A; the second should show the column space (or range) of A.
   (c) For a full-rank A, make a schematic diagram (sketch) showing the image of the unit circle $x^2 + y^2 = 1$ under the transform $A \begin{bmatrix} x \\ y \end{bmatrix}$.
   (d) Give the definition of the matrix 2-norm $\|A\|_2$. Draw a diagram illustrating the definition.
   (e) Consider s as the set of all points on a line segment. What is the image of s under transformation by A? Justify your answer.
   (f) Sketch the set of points (x, y) in $\mathbb{R}^2$ where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ has $\|\vec{v}\|_1 = 1$.
   (g) Sketch the image of your previous answer under transformation by A. Indicate $\|A\|_1$ on your sketch.
3. (12 points) Recall that an $n \times n$ *orthogonal matrix* Q is a matrix where the columns form an orthonormal basis of $\mathbb{R}^n$.
   (a) What is $Q^T Q$? Justify your answer.
   (b) What is Q^{-1}?
   (c) Suppose we are given a vector b, an orthogonal matrix Q, and an upper triangular matrix R, such that $A = QR$. Describe a simple way to solve $Ax = b$ (assuming such an x exists) without explicitly forming A and without using Gaussian Elimination.
   (d) Continue to assume that $A = QR$ is as in (c), and define $B = RQ$. What can be said about the eigenvalues of A and the eigenvalues of B? Justify your answer.

4. (16 points) Consider the differential equation

$$2y'' + ay' + 18y = F\cos(\omega t).$$

   (a) Suppose $a = 12, F = 0$. Find the general solution to the ODE.
   (b) Now suppose that $a = 0$. At what value of ω will there be resonance?
   (c) Solve the differential equation in the case $a = 0, F = 3$ and ω as in part (b) (the resonance case) given the initial conditions $y(0) = 2, y'(0) = 0$.

5. (14 points) Create a phase plane diagram of the following system of equations

$$\frac{dx}{dt} = y(1 - y - x^2),$$
$$\frac{dy}{dt} = x(y - 2).$$

   in the region $-\infty < x < \infty, -\infty < y < \infty$.

   Your diagram should clearly indicate the number and types of steady states. You should support the diagram with calculations of stability properties of the steady states.

6. (18 points) The following model has been used to describe population densities $u^+(x,t), u^-(x,t)$, of individuals moving along the x axis:

$$\frac{\partial u^+}{\partial t} + v\frac{\partial u^+}{\partial x} = -\lambda^+ u^+ + \lambda^- u^-, \tag{1a}$$
$$\frac{\partial u^-}{\partial t} - v\frac{\partial u^-}{\partial x} = \lambda^+ u^+ - \lambda^- u^-. \tag{1b}$$

   where $v > 0, \lambda^+ > 0, \lambda^- > 0$ are all constant.

   (a) Which density describes those moving right? left? Explain your answer. What is the possible meaning of terms multiplied by λ^+, and λ^-?
   (b) In a paper on this subject, the authors used the domain $0 \le x \le \ell$ and a set of boundary conditions for this problem. Which of the following Dirichlet conditions would make sense? Explain your answer
      (i) $u^+(0,t) = 0, u^-(0,t) = 0$
      (ii) $u^+(\ell,t) = 1, u^-(\ell,t) = 1$
      (iii) $u^+(0,t) = 0, u^-(\ell,t) = 0$
      (iv) $u^+(\ell,t) = 1, u^-(0,t) = 1$
   (c) Now let $\lambda^+ = \lambda^- = \lambda > 0$. Define $u(x,t) = u^+(x,t) + u^-(x,t)$ and $w(x,t) = u^+(x,t) - u^-(x,t)$. Show that the above model can be rewritten in terms of u, w as follows:

$$\frac{\partial u}{\partial t} = -v\frac{\partial w}{\partial x} \tag{2a}$$
$$\frac{\partial w}{\partial t} = -v\frac{\partial u}{\partial x} - 2\lambda w \tag{2b}$$

      (Hint: add and subtract the original equations.)
   (d) Show that your system (2) can be rewritten in the form of the telegraph equation,

$$\frac{\partial^2 w}{\partial t^2} + 2\lambda\frac{\partial w}{\partial t} = v^2\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

      (Hint: differentiate one equation in the system (2) with respect to t, the other with respect to x and eliminate a term $\partial^2 u/\partial x\partial t$.) Note that w can be positive or negative, based on its definition.

(e) Use separation of variables to solve Equation (3) subject to a new set of boundary conditions (not the same as in part (b)!)

$$w(0,t) = w(\ell,t) = 0$$

for arbitrary initial conditions. NOTE: you can do this part even if you did not solve parts (a)-(d).

(f) Explain the behaviour of your solution $w(x,t)$ in part (e)- whether it decays or grows in time, and under what conditions it has decaying or growing oscillations.