# The University of British Columbia
## Department of Mathematics
## Qualifying Examination—Differential Equations
January 2020

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1. (15 points) Consider the following problem for the two-component vector $\mathbf{y}(t)$ satisfying the linear system of ODEs
$$\mathbf{y}'' + \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} \mathbf{y} = F\cos(\omega t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Here $F > 0$ and $\omega > 0$ are constants.
   (a) Determine in as explicit a form as you can the general solution to the homogeneous problem where $F = 0$.
   (b) Calculate a particular solution for $F > 0$ and for any $\omega$ with $\omega \neq \sqrt{3}$ and $\omega \neq 2$.
   (c) Calculate a particular solution for $F > 0$ and $\omega = 2$. (Hint: this is a resonance case).
   (d) When $\omega = 3$ and $F = 1$, find an explicit solution to the initial value problem with initial condition
$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{y}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2. (15 points) Let $c_0$ and $c$ be positive constants with $0 < c_0 < c$. Calculate the solution $u(x,t)$ to each of the following two PDE wave problems:
   (a) The one-way wave equation:
$$u_t + c\,u_x = 0\,, \quad c_0 t \leq x < \infty\,, \quad t \geq 0\,,$$
$$u(x,0) = f(x)\,, \qquad u(c_0 t, t) = h(t)\,.$$
   (b) The wave equation:
$$u_{tt} = c^2\,u_{xx}\,, \qquad c_0 t \leq x < \infty\,, \quad t \geq 0\,,$$
$$u(x,0) = f(x)\,, \qquad u_t(x,0) = 0\,, \qquad u(c_0 t, t) = h(t)\,.$$
   (c) Briefly explain (three clear sentences is sufficient here) the qualitative differences between the solutions to these two problems.

3. (15 points) Let $u(r,\theta)$ satisfy Laplace's equation in polar coordinates inside the disk $0 \leq r \leq R$ with $0 \leq \theta \leq 2\pi$, satisfying
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \quad 0 \leq r \leq R\,, \quad 0 \leq \theta \leq 2\pi\,,$$
$$u(R,\theta) = f(\theta)\,; \quad u \text{ bounded as } r \to 0\,; \quad u\,,\ u_\theta\,, \quad 2\pi \text{ periodic in } \theta\,.$$
   (a) Determine an explicit solution for $u$ when $f(\theta) = 2\cos^2(2\theta)$.
   (b) For an arbitrary smooth $2\pi$ periodic $f(\theta)$ derive an infinite series representation for $u(r,\theta)$ in terms of the angular eigenfunctions.
   (c) Sum the infinite series in (ii) to obtain the well-known Poisson's integral formula for $u(r,\theta)$. (Hint: you will need to calculate an infinite sum of the form $\sum_{n=1}^{\infty} \rho^n \cos(n\theta)$ for $0 < \rho < 1$.)

4. (15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

   (a) (3 points) What is the rank of $A$?

   (b) (4 points) Find a basis for the nullspace of $A^T$ (the transpose of $A$).

   (c) (6 points) Determine all values of $w$ for which the system

$$x + y = -1 \tag{10}$$
$$x + 2y = w \tag{11}$$
$$3x + 4y = 0 \tag{12}$$

   has a solution and find *one*.

   (d) (2 points) Is the solution in (c) above unique?

5. (15 points) Consider the sequence $\{\mathbf{v}^n\}_{n=0}^{\infty}$ of vectors in $\mathbb{R}^2$ defined by given $\mathbf{v}^0$ and the recurrence relationship

$$A\mathbf{v}^{n+1} = B\mathbf{v}^n$$

where

$$A = \begin{bmatrix} 1 & -k \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$$

and $k > 0$ is a given parameter.

   (a) (8 points) Rewite the recurrence relationship in the form

$$\mathbf{v}^{n+1} = C\mathbf{v}^n$$

   with $C$ a matrix.

   (b) (7 points) Show that $\|\mathbf{v}^n\| = \|\mathbf{v}^0\|$ for all $n$ where $\|\cdot\|$ is the standard Euclidean norm.

6. (15 points) (a) (7 points) Prove that

$$e^A := I + A + A^2/2! + \cdots + A^m/m! + \ldots$$

   converges at every index for any square matrix. Here, $I$ is the identity matrix.

   (b) (4 points) Find $e^A$ when

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

   (c) (4 points) Find $e^A$ when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$