The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September 2021

Real analysis

1. (10 points) Let $\mathbf{F}$ be the vector field $(x + e^{y^2})\,\hat{\boldsymbol{\imath}} + (y - \sin(z^2))\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\mathbf{k}}$, and let S be the boundary of the region

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4, 0 \leq z \leq 1\},$$

oriented so that the normal points outwards. Calculate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

2. (10 points) Determine if the following assertions are **true** or **false**, justifying the answers carefully.
   (a) Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be real numbers. If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.
   (b) Let $(a_n)_{n=1}^{\infty}$ be real numbers such that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then the sequence of functions $f_N(x) = \sum_{n=1}^{N} a_n e^{x^n}$ defined on $[-1, 1]$ converges uniformly as $N \to \infty$.
   (c) If the sequence of continuously differentiable functions $f_n : [0, 1] \to \mathbb{R}$ converges uniformly to f, then f must be differentiable on all of $[0, 1]$.

3. (10 points) (a) (3 points) Let $D = \{(x, y) \in \mathbb{R}^2 : xy \in \mathbb{Q}\}$. Prove that both D and $\mathbb{R}^2 \setminus D$ are dense in $\mathbb{R}^2$. You may use the fact that rationals and irrationals are dense in $\mathbb{R}$.
   (b) (7 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} x & \text{if} \quad xy \in \mathbb{Q} \\ y & \text{if} \quad xy \notin \mathbb{Q} \end{cases}.$$

   Find all the points $(a, b) \in \mathbb{R}^2$ such that f is continuous at (a, b).

Complex analysis

4. (10 points) Compute the contour integral

$$\oint_C \frac{z+1}{z^3+2z^2}\, dz,$$

where C denotes

(a) (5 points) the circle $\{z : |z| = 1\}$ traversed once in the counterclockwise direction.

(b) (5 points) the circle $\{z : |z + 2 - i| = 2\}$ traversed once in the counterclockwise direction.

5. (10 points) (a) (7 points) Find all singularities of the function

$$f(z) = \frac{z^3}{1 - \cos(z^2)}.$$

Determine the nature of each singularity (i.e., whether it is removable, essential or a pole). For each pole, determine its order.

(b) (3 points) Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 3$ and $|f(z)| \le 8\,|e^z|$ for all $z \in \mathbb{C}$.

6. (10 points) (a) (5 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $A, B > 0$ be positive *real* numbers such that

$$|f(z)| \le A\,|z| + B \quad \text{for all } z \in \mathbb{C}.$$

Show that there exists $a, b \in \mathbb{C}$ such that

$$f(z) = az + b \quad \text{for all } z \in \mathbb{C}.$$

Hint: Show that f'' is identically zero.

(b) (5 points) Find the number of zeros (where each zero is counted as many times as its multiplicity) of the polynomial $f(z) = z^6 - 5z^2 + 10$ in the annulus $A = \{z : 1 < |z| < 2\}$.
Hint: Apply Rouché's theorem.