Analysis Qualifying Exam

University of British Columbia

August 30th 2013

1. Suppose that $\{a_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} a_n = L$. Prove that
$$\lim_{n\to\infty} \frac{1}{n}(a_1 + \cdots + a_n) = L.$$

2. Find the point (or points, if there is more than one) on the surface $xyz^3 = 2$ that is closest to the origin.

3. Use Stokes's Theorem to evaluate $\int_C (x-z)dx + (x+y)dy + (y+z)dz$, if C is the ellipse where the plane $z = y$ intersects the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

4. (a) Use residues to compute
$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2 + 4x + 5} dx\,.$$

   (b) How many roots (counted with multiplicity) of the equation $z^4 - 5z^3 + z - 2 = 0$ lie in the disk $|z| < 1$?

5. Let $n \geq 1$ and let $\{a_0, a_1, \ldots, a_n\}$ be complex numbers such that $a_n \neq 0$. For $\theta \in \mathbb{R}$ define
$$f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \cdots + a_n e^{ni\theta}\,.$$
Prove that there exists some $\theta \in \mathbb{R}$ such that $|f(\theta)| > |a_0|$.

6. (a) Let a be a complex number and let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function such that $f(az) = f(z)$ for all $z \in \mathbb{C}$. Prove that there exists a positive integer n such that $a^n = 1$.

   (b) Let a and b be complex numbers and let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function such that $f(az + b) = f(z)$ for all $z \in \mathbb{C}$. Prove that there exists a positive integer n such that $a^n = 1$.