Pure Math Qualifying Exam: Jan. 8, 2005

Part I

1. Let b and c be real numbers, $c > 0$. Use contour integration to evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\cos(x-b)}{x^2+c^2} dx.$$

2. Let A be an $n \times n$ real symmetric matrix, and define the matrix e^A by the convergent series
$$e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \cdots .$$
Show that e^A is non-singular.

3. Let p be a prime number and G a group of order p^3. Show that for any $g, h \in G$
$$g^p h = h g^p.$$

4. Consider the vector field
$$\mathbf{F}(x,y,z) = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}.$$
(a) Verify that $\nabla \cdot \mathbf{F} = 0$ on $\mathbb{R}^3 \backslash \{0\}$.
(b) Let S be a sphere centred at the origin, with "outward" orientation. Show that
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi. \qquad (*)$$
(c) Now let $E \subset \mathbb{R}^3$ be an open region with smooth boundary S (given "outward" orientation), and suppose $0 \in E$. Show that $(*)$ still holds.

5. (a) Let f(z) be an analytic function on an open, bounded, connected region $\Omega \subset \mathbb{C}$, and suppose f is continuous on the boundary $\partial\Omega$. Suppose also that $|f(z)|$ is constant on $\partial\Omega$. Show that either f has a zero in Ω, or f is constant.
(b) Find all functions which are analytic in the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$, continuous on the closed unit disk, and which satisfy $|z| \le |f(z)| \le 1$ for $|z| \le 1$.

6. Let V be the vector space of all polynomials $p(x)$ with real coefficients. Let A and B denote the linear transformations on V of (respectively) multiplication by x, and differentiation. That is, $A : p(x) \mapsto xp(x)$, and $B : p(x) \mapsto p'(x)$.
(a) Show that A has no eigenvalues, and that 0 is the only eigenvalue of B.
(b) Compute the transformation $BA - AB$.
(c) Show that no two linear transformations A, B on a *finite dimensional* real vector space can satisfy $BA - AB = I$ (here I denotes the identity transformation).

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Part II

1. (a) Show that a continuous function on $\mathbb{R}$ cannot take every real value exactly twice.

   (b) Find a continuous function on $\mathbb{R}$ that takes each real value exactly 3 times.

2. Let R be the ring
$$R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z} + \mathbb{Z}\sqrt{-3}.$$

   (a) Show that $2R \subset R$ is not a prime ideal.

   (b) Show that 2 is an irreducible element of R; i.e., if $2 = uv$ then either u or v is a unit.

   (c) Is R a principal ideal domain (PID)?

3. Show that an entire function $f(z)$ satisfying $\lim_{|z|\to\infty} |f(z)| = c$ (for some $c \in (0, \infty)$) is constant.

4. For a vector $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$, define $\|v\|_1 := \sum_{j=1}^n |v_j|$, and for an $n \times n$ matrix A, define
$$\|A\|_1 := \sup_{v\in\mathbb{R}^n; v\neq 0} \frac{\|Av\|_1}{\|v\|_1}.$$
Show that if $A = (a_{ij})$, then
$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|.$$

5. Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous functions on $[0, 1]$ satisfying
$$|f_n(x) - f_n(y)| \leq L|x - y|$$
for all $x, y \in [0, 1]$, and for all n (here L is a fixed constant), and suppose f_n converges pointwise to a function f. Show that f_n converges to f uniformly, and that
$$|f(x) - f(y)| \leq L|x - y|$$
for all x, y.

6. Let $\alpha \in \mathbb{C}$ be an algebraic number and p a prime. Show that there exist field extensions of finite degree
$$\mathbb{Q} \subset F \subset K,$$
such that $\alpha \in K$, the degree $|K : F|$ is a power of p and $|F : \mathbb{Q}|$ is prime to p.