Winter 2008, Pure Qualifying Exam

Part 1

1. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative real numbers and that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

2. The periodic function $f(x)$ is defined by

$$f(x) = e^x, \quad \text{for } -\pi \leq x \leq \pi \text{ and } f(x+2\pi) = f(x).$$

Find the Fourier series representation of $f(x)$. Check whether the series can be differentiated to give the familiar result,

$$\frac{d}{dx}(e^x) = e^x.$$

Relate your results to the continuity properties of $f(x)$.

3. (a) State the divergence theorem, and use it to evaluate the integral,

$$\int\int_S \mathbf{u} \cdot \mathbf{n} \, ds$$

where

$$\mathbf{u} = (xz^2, \sin x, y)$$

and S is the closed surface of the cylinder bounded by

$$x^2 + y^2 = 1, \;\; z = 0, \;\; z = 2.$$

What would the result have been had $\mathbf{u} = \nabla \times \mathbf{a}$ for any differentiable vector field $\mathbf{a}(\mathbf{x})$?

(b) State Stokes' theorem, and use it to evaluate the integral

$$\int_C \mathbf{u} \cdot \mathbf{dr}$$

where C is the unit circle $x^2 + y^2 = 1$, directed in an an anticlockwise sense, and

$$\mathbf{u} = (\cos x, 2x + y \sin y, x)$$

What would the result have been had $\mathbf{u} = \nabla\phi$ for any differentiable scalar field, $\phi(\mathbf{x})$?

4. *(i)* Prove that an orthogonal set of vectors $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ in an $n-$dimensional Euclidean space is linearly independent.

(ii) Let V be a subspace of $\Re^4$ spanned by the vectors,

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

Using the Gram-Schmidt procedure, construct an orthogonal basis for V.

(iii) Consider the vector space formed by all polynomials, $P_n(x)$ with $-1 \leq x \leq 1$, of degree less than or equal to n. Consider the inner product,

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Determine the quadratic polynomial, $P_2(x)$, which is normalized so that $P_2(0) = 1$ and is orthogonal to both polynomials $P_0(x) = 1$ and $P_1(x) = x$.

5. (i) Define a Hermitian matrix, and prove that all of its eigenvalues are real, and that the eigenvectors corresponding to distinct eigenvalues are orthognal.

(ii) Find a matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 4 & 2 \\ -1 & -3 & 0 \end{pmatrix}.$$

6. Let A be an $n \times m$ matrix. Prove that the equation $Ax = b$ has a solution if and only if $\langle b, v \rangle = 0$ for all v in the nullspace of A^*.

Part 2

1. Using contour integration, find the definite integrals

$$(a) \qquad \int_0^\infty \frac{\ln x}{x^2+1} dx$$

and

$$(b) \qquad \int_{-\infty}^\infty \frac{e^{ikx}}{\cosh x} dx$$

with k a parameter (Hint: for (b) use a rectangular contour).

2. Find all possible Laurent expansions of

$$\frac{1}{(2+z)(z^2+1)}$$

about $z = 0$.

3. Consider the annular region, D, given by $\frac{1}{5} \leq |z| \leq 1$.

(i) Show that

$$\phi(z) = 2 + \frac{\ln |z|}{\ln 5}$$

is harmonic in D.

(ii) Show that

$$w = f(z) = \frac{3z+1}{3+z}$$

is a conformal mapping of D. Show that the image, E, of D in the $w-$plane is bounded by two non-concentric circles, C_1 and C_2, with C_1 contained inside C_2.

(iii) Suppose that $\Phi(w)$ is harmonic on E such that $\Phi = 2$ on C_2 and $\Phi = 1$ on C_1. Find $\Phi(w)$.

4. Show that a unique factorization domain is a principal ideal domain if and only if every non-zero prime ideal is maximal.

5. Let F be a field and let α_i be elements of f. Suppose that $f(x) = \prod_{i=1}^n (x - \alpha_i) \in F[x]$ is a polynomial. Recall that the *discriminant* of f is

$$D(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

(i) Suppose $f(x) = x^3 + ax + b \in F[x]$. Show that

$$D(f) = -4a^3 - 27b^2.$$

(Hint: When viewed as a function of the α_i, D is homogeneous.)

(ii) Show that the polynomial

$$f(x) = x^3 - 48x + 64$$

is irreducible over $\mathbb{Q}$.

(iii) Compute the Galois group over $\mathbb{Q}$ of $x^3 - 48x + 64$.

6. Let G be a finite group and let H be proper subgroup. Show that G is not equal to the union of the conjugates of H.