Mathematics Qualifying Exam
University of British Columbia
January 14, 2012

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. (a) Find all polynomials that are uniformly continuous on $\mathbb{R}$.

   (b) Let A be a nonempty subset of $\mathbb{R}$ and let f be a real-valued function defined on A. Further let $\{f_n\}$ be a sequence of bounded functions on A which converge uniformly to f. Prove that

   $$\frac{f_1(x) + \cdots + f_n(x)}{n} \to f(x)$$

   uniformly on A as $n \to \infty$.

2. (a) Prove the Logarithmic Test

   Theorem 1. *Suppose that $a_k \neq 0$ for large k and that*

   $$p = \lim_{k\to\infty} \frac{\log(1/|a_k|)}{\log k} \text{ exists.}$$

   - *If $p > 1$ then $\sum_{k=1}^{\infty} a_k$ converges absolutely, and*
   - *If $p < 1$ then $\sum_{k=1}^{\infty} |a_k|$ diverges.*

   (b) Let $\{a_k\}$ be a sequence of non-zero real numbers and suppose that

   $$p = \lim_{k\to\infty} k\left(1 - \left|\frac{a_{k+1}}{a_k}\right|\right) \text{ exists}$$

   Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely when $p > 1$.

3. Evaluate the integral

   $$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma,$$

   where S is the region of the plane $y = z$ lying inside the unit ball centred at the origin, and $\mathbf{F} = (xy, xz, -yz)$, and $\mathbf{n}$ is the upward-pointing normal.

   Note that it might be helpful to remember that

   $$\int 2\sin^2 t dt = t - \sin t \cos t.$$

4. In the following, justify your answer.

   (a) (6 points) Prove or disprove:
   There exists a holomorphic function f on $\mathbb{C}$ (thus an entire function) such that $f(D) = Q$ where D is the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and Q is the square $Q = \{z \in \mathbb{C} \mid -1 < \operatorname{Re} z, \operatorname{Im} z < 1\}$.

   (b) (7 points) Find all holomorphic functions $f(z)$ on $\mathbb{C} \setminus \{0\}$ such that

$$f(1) = 1, \qquad |f(z)| \leq \frac{1}{|z|^3}$$

   (c) (7 points) Find a holomorphic function $f(z)$ on $D = \{z \in \mathbb{C} \mid |z| < 1\}$, which maps D onto the infinite sector

$$S = \{z = re^{i\theta} \in \mathbb{C} \mid 0 < \theta < \pi/4\}.$$

5. (a) (6 points) Prove or disprove:
   There exists a **nonconstant** holomorphic function $f(z)$ from $D = \{z \in \mathbb{C} \mid |z| < 1\}$ into $\mathbb{C}$ such that the area of its image, area $f(D) = 0$.

   (b) (7 points) Show that there is **no** holomorphic function $f(z)$ on $D = \{z \in \mathbb{C} \mid |z| < 1\}$ such that $|f(z)| = |z|^{1/2}$ for all $z \in D$.

   (c) (7 points) Find all harmonic functions $u(x, y)$ on $\mathbb{R}^2$ such that $e^{u(x,y)} \leq 10 + (x^2 + y^2)$ and $u(1, 1) = 0$.

6. (20 points) Evaluate the following integral, using contour integration, carefully justifying each step:

$$\int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$$

Linear Algebra

1. Determine the eigenvalues and a basis of the corresponding eigenspaces for the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $\mathbf{A}$ with respect to the standard basis, where:

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

   Note: all eigenvalues are rational numbers.

2. Let $\mathcal{N}_n \subset M_n(\mathbb{R})$ be the set of *nilpotent* matrices, that is the set of $n \times n$ matrices A such that $A^k = 0$ for some k. Show that $\mathcal{N}_n$ is a closed subset of $M_n(\mathbb{R})$ (identify the latter with $\mathbb{R}^{n^2}$).

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
   (a) Show that there is a unique integer $0 \leq k \leq \min\{n, m\}$ for which there are bases $\{\underline{u}_i\}_{i=1}^n \subset \mathbb{R}^n$ $\{\underline{v}_i\}_{i=1}^m \subset \mathbb{R}^m$ such that the matrix of T with respect to these bases is $D^{(k)}$, where

$$D^{(k)} = \begin{cases} 1 & 1 \leq i = j \leq k \\ 0 & \text{otherewise} \end{cases},$$

   that is $D^{(k)}$ has zeroes everywhere except that the first k entries on the main diagonal are 1.
   (b) Show that the row rank and column rank of any matrix $A \in M_{m,n}(\mathbb{R})$ are equal.

Differential Equations

1. Consider the differential equation
$$4x^2 \frac{d^2y}{dx^2} + y = 0.$$
   (a) For $x > 0$ find all solutions $y(x)$.
   (Hint: look for solutions of the form $y(x) = \sqrt{x} f(x)$.)

   (b) Determine $y(x)$ in the limit $x \to +0$.

2. The following system of differential equations:
$$\frac{dx_1}{dt} = 2x_1 - x_2 + t$$
$$\frac{dx_2}{dt} = 3x_1 - 2x_2$$
has a linear solution. Determine the set of all solutions $(x_1(t), x_2(t))$.

3. Consider the initial value problem
$$u_{tt} - u_{xx} = f(x) \cos t$$
$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad -\infty < x < \infty, 0 \le t < \infty$$
for a continuous function $f(x)$ on $\mathbb{R}$, which vanishes for $|x| > R$.

   (a) Solve the initial value problem.
   Note: The solution is of the form $u(x,t) = u_p(x,t) + u_h(x,t)$. Use separation of variables to find a particular solution $u_p(x,t)$ of $u_{tt} - u_{xx} = f(x) \cos t$ (ignoring the initial values). Then, $u_h(x,t)$ is a solution to the homogenous PDE with appropriately adjusted initial conditions.

   (b) The particular solution $u_p(x,t)$ is not unique. Because of that it is not obvious whether the solution $u(x,t)$ is unique. Prove that it is.