Algebra Qualifying Exam

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1. Let S_1 and S_2 be subspaces of a vector space.
   (a) Give an example to show that in general $S_1 \cup S_2$ need not be subspace.
   (b) If $S_1 \cup S_2$ is a subspace, prove that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

2. Let V be an n-dimensional linear subspace of $\mathbb{R}^N$ ($1 \leq n \leq N$). We denote by $\langle v, w \rangle$ the ordinary dot product of vectors $v, w \in \mathbb{R}^N$.
   (a) If $\{v_1, \ldots, v_n\}$ is a basis for V, define for each i the linear map $\nu^i(v_j) = \delta^i_j$ where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_i = 1$. This definition on the basis vectors then uniquely determines ν^i as an element of V^*, the dual space of V. Prove that $\{\nu^1, \ldots, \nu^n\}$ is a basis for V^*.
   (b) Given $v \in V$, let $v^* : V \to \mathbb{R}$ be the linear map $v^*(w) = \langle w, v \rangle$. Show that the map $v \to v^*$ defines a linear isomorphism from V to V^*.
   (c) We extend the dot product to V^* by requiring $\langle v^*, w^* \rangle = \langle v, w \rangle$ for $v, w \in V$. If $\{v_1, \ldots, v_n\}$ is a basis for V, let $g_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$. Show that $\langle \nu^i, \nu^j \rangle = g^{ij}$ where the matrix (g^{ij}) is the inverse matrix of (g_{ij}).
   (d) Determine the change of basis matrix from $\{v_1^*, \ldots, v_n^*\}$ to $\{\nu^1, \ldots, \nu^n\}$.

3. (a) Let A be an $n \times n$ real matrix. Show that if $A^k = 0$ for some positive integer k and A is not the zero matrix, then A cannot be diagonalized. Show that 0 is an eigenvalue of A.
   (b) Let V be a vector space, and suppose that $v_1, \ldots, v_n$ are eigenvectors of a linear map $T : V \to V$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $v_1, \ldots, v_n$ are linearly independent.

4. Let R be a commutative ring with identity and let $I, J \subset R$ be two ideals. Define the *colon ideal*
$$(I : J) = \{r \in R \mid rj \in I \text{ for all } j \in J\}.$$
   You may assume without proof that this is indeed an ideal in R.
   (a) Let $R = \mathbb{Z}[x]$ and $I = (f(x))$, $J = (g(x))$ for some polynomials $f(x), g(x) \in \mathbb{Z}[x]$. Show that $(I : J)$ is principal and describe its generator in terms of irreducible factors of $f(x)$ and $g(x)$.
   (b) Let $R = \mathbb{Z}[x]$, $I = (x - 2, 6)$ and $J = (x)$. Find a finite set of generators for the ideal $(I : J)$.

5. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 7 and let E be the splitting field of $f(x)$ over $\mathbb{Q}$. Assume that $Gal(E/\mathbb{Q}) = S_7$.

(a) Find the number of intermediate fields K between $\mathbb{Q}$ and E, such that

$$\deg(E/K) = 9.$$

(b) Show that the intersection of all fields K in part (a) is not equal to $\mathbb{Q}$.

(c) If $\alpha \in E$ is a root of $f(x)$, how many of the intermediate fields in part (a) contain α?

6. In each case below, find the Galois group of the polynomial $f(x)$ over $\mathbb{Q}$. Describe the Galois group as an abstract group and also give its action on the roots of $f(x)$.

   (a) $f(x) = x^4 + 4$.

   (b) $f(x) = x^6 - 2$.