Qualifying Exam Problems: Differential Equations

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1. (10 points) Let $A \in M_{n,n}(\mathbb{R})$ be a matrix of rank $n-1$. Let $L_A : M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$ be the function given by $L_A(B) = A \cdot B$.
   (a) Show that L_A is a linear map.
   (b) Find the dimension of the image of L_A.
   (c) Find a basis for the image of L_A.

2. (10 points) Let $k \in \mathbb{N}$, let $A_1, \ldots, A_k \in M_{n,n}(\mathbb{R})$ and let

$$B = \sum_{i=1}^{k} A_i \cdot A_i^t,$$

   where for each matrix C, we denote by C^t its transpose.
   (a) Prove that B is a symmetric matrix.
   (b) Prove that B is a positive definite matrix, i.e. for each vector $v \in M_{n,1}(\mathbb{R})$, the dot product $\langle Bv, v\rangle$ is nonnegative.
   (c) Prove that $\det(B) \geq 0$.

3. (10 points) Solve the following system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 0 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 & = 2 \\ -2x_1 - x_2 + x_3 + 2x_4 & = 1 \\ -10x_1 - 8x_2 - 4x_3 - 2x_4 & = 2 \end{cases}.$$

   Explain your answer.

4. (10 points) Find the solution $y(t)$ of $y'' + y' - 2y = 4te^{2t}$ satisfying $y(0) = y'(0) = 0$.

5. (10 points) Consider the autonomous system

$$\dot{x} = 2y - xf(x,y)$$
$$\dot{y} = -x - yf(x,y)$$

   where $f(x,y)$ is a smooth real valued function on the plane.
   (a) Find the critical point(s) of the system.
   (b) Find a function $f(x,y)$ so that all solutions $x(t), y(t)$ are bounded for all t, but do not all converge to a critical point as $t \to \infty$.
   (c) Determine the long time behaviour of solutions if $f(x,y) > 0$.
   (d) Find an example of a function $f(x,y)$ and an initial condition so that the solution $x(t), y(t)$ blows up in finite time.

6. (10 points) Consider the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0$$

   on the interval $0 \leq x \leq 1$, with boundary conditions

$$y(0) = 0$$
$$y(1) + y'(1) = 0.$$

   Let $\lambda_1 \leq \lambda_2, \leq \lambda_3 \cdots$ be the eigenvalues listed in increasing order.

(a) Show that $\lambda_1 > 0$.

(b) Write down the equation that determines the eigenvalues and give a qualitative description of the large n behaviour of λ_n.

(c) Determine the eigenfunctions $\varphi_n(x)$, normalized so that $\int_0^1 \varphi_n(x)^2 dx = 1$, in terms of the eigenvalues you found in the previous part.

(d) Use an eigenfunction expansion to solve

$$u_t(x,t) = u_{xx}(x,t) + t$$

for $0 \leq x \leq 1$ and $t \geq 0$ where

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) + u_x(1,t) &= 0 \\ u(x,0) &= 0 \end{aligned}$$