Mathematics Qualifying Exam
University of British Columbia
September 2, 2010

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. a) Find all continuously differentiable vector fields $\mathbf{F}(x,y,z)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ with the property that $\displaystyle\oint_C \mathbf{F}(x,y,z) \times d\mathbf{r} = \mathbf{0}$ for all closed curves C in $\mathbb{R}^3$.

   b) Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

2. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbb{R}^n$ is the n-dimensional Euclidean space with the standard Euclidean metric. Suppose that for all k large enough, T^k satisfies:

$$|T^k x - T^k y| \le r|x-y|$$

   where $r \in (0,1)$. Show that T has a unique fixed point.

3. a) Show that $I = \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

   b) Evaluate I.

   c) For which real values of p and q does $\displaystyle\int_0^\infty \frac{\sin x}{|1-x|^q\, x^p}\,dx$ converge? Justify your answer.

4. Consider the function $f(z) = e^{\frac{z+1}{z-1}}$. Find all z for which

   a) $|f(z)| = 1$;

   b) $|f(z)| < 1$.

5. Let $f(z) = \dfrac{1}{(2z-1)(z-2)}$.

   a) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about $\frac{1}{2}$.

   b) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about 0, valid for small $|z|$. Give the region of convergence for the full expansion.

   c) Give the first three nonzero terms for the Laurent expansion of $f(z)$ about 0, valid for large $|z|$. Give the region of convergence for the full expansion

   d) Compute $f''(0)$.

   e) Evaluate $\oint_{|z|=\frac{3}{2}} f(z)\,dz$.

   f) Evaluate $\oint_{|z|=79} f(z)\,dz$.

6. Let H^+ be the upper half plane. For $z \in \mathsf{H}^+$, define the function

$$h_z(\zeta) = \frac{1}{2\pi i}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar{z}}\right) \quad \text{for all } \zeta \in \mathsf{H}^+,$$

   then h_z is analytic on H^+ except for a simple pole at z. Suppose that f is a bounded analytic function on $\mathsf{H}^+ \cup \mathbb{R}$. Prove that

$$\int_{-\infty}^{\infty} f(t)\,h_z(t)\,dt = f(z).$$

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Part II: Linear Algebra and Differential Equations (Applied Exam)

1. A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite if $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Show:

   a) If $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, then all its eigenvalues are positive.

   b) If $A \in \mathbb{R}^{n\times n}$ is positive definite and $X \in \mathbb{R}^{n\times k}$ has rank k, then $X^\top AX$ is also positive definite.

   c) Let A be the symmetric matrix

   $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$

   Verify that if A is positive definite, then

   $$a_{11} > 0, \qquad a_{22} > 0, \qquad |a_{12}| < \frac{a_{11} + a_{22}}{2}.$$

2. The trace of a matrix $A \in \mathbb{R}^{n\times n}$ is defined by $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

   a) Show that if $A \in \mathbb{R}^{n\times n}$ and $A' \in \mathbb{R}^{n\times n}$ are similar, then $\mathrm{tr}(A) = \mathrm{tr}(A')$.

   b) For any matrix $A \in \mathbb{R}^{n\times n}$, there holds $\det(e^A) = e^{\mathrm{tr}(A)}$. Prove this identity in the case where A is diagonalizable.

3. Let A be a square matrix with all diagonal entries equal to 2, all entries directly above or below the main diagonal equal to 1, and all other entries equal to 0. Show that every eigenvalue of A is a real number strictly between 0 and 4.

Please turn over

4. Consider the ODE

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \sin^2 t, \qquad b = \text{const},\ c = \text{const}. \tag{1}$$

a) For which values of the constants b and c does (1) have a periodic solution, i.e., (1) has a particular solution $y_p(t)$ with the property that $y_p(t) = y_p(t+T)$ for some constant T?

b) Find the period of the periodic solution when a periodic solution of (1) exists, i.e., find the smallest T with the property that $y_p(t) = y_p(t+T)$.

c) Consider the situation when (1) has a periodic solution $y_p(t)$. Let $y(t) = Y(t)$ be the solution of the initial value problem

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \sin^2 t, \qquad y(0) = 79.333333336,\ y'(0) = 2.2222221.$$

For which values of b and c is it true that $\lim_{t\to\infty} |Y(t) - y_p(t)| = 0$?

5. Solve the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= 2x - y, \\ \frac{dy}{dt} &= 4x - 3y + e^t, \end{aligned}$$

where $x(0) = 1,\ y(0) = 0$.

6. a) Bacteria growth can be modelled by the reaction-diffusion equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + \beta^2 u, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= f(x), \end{aligned}$$

where $u(x,t)$ is the bacteria concentration, α^2 is the diffusion rate, β^2 the growth rate, and $f(x)$ a given initial concentration.

a1) Find all values of the constants α and β for which the bacteria concentration will not decay to zero as $t \to \infty$.

a2) Find all values of the constants α and β for which $\lim_{t\to\infty} u(x,t) = 0$ for any initial datum $f(x)$.

b) The temperature $u(x,t)$ in a one-dimensional rod satisfies the heat equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u(x,0) &= f(x), \end{aligned}$$

for a given initial temperature distribution $f(x)$ and a thermal diffusivity parameter α^2. Determine the steady-state temperature $\lim_{t\to\infty} u(x,t)$ for the following boundary conditions:

b1) $u(0,t) = T_1$ and $u(L,t) = T_2$ for given constants T_1 and T_2.

b2) $u_x(0,t) = u_x(L,t) = 0$.