Mathematics Qualifying Exam
University of British Columbia
September 2, 2011

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. a) Find the closest points to the origin on the ellipse $C : x^2 + 4y^2 + 2x = 3$.

   b) Compute $\oint_C y^2 dx + x dy$, where C is the above same ellipse oriented counterclockwise.

2. The Arzelà-Ascoli theorem states: Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of real-valued continuous functions defined on an interval I of the real line. If the interval I is bounded and this sequence is uniformly bounded and equicontinuous, then there exists a subsequence (f_{n_k}) that converges uniformly on I. Show, by way of example,

   a) the necessity of the uniform boundedness,

   b) the necessity of the equicontinuity, and

   c) the necessity of the boundedness of I.

   For all the above, explain why your examples do not converge uniformly.

3. a) Show the convergence of the improper integral $\quad C_1 = \displaystyle\int_0^\infty \frac{\sin t}{t}\, dt.$

   b) Assume $f(t) : [0,1] \to \mathbf{R}$ is a smooth function with $f(0) = 1$. Find the value of $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{\sin nt}{t} f(t)\, dt$. You may assume part a).

Please turn over

4. a) Let $f(x,y) = u(x,y) + iv(x,y)$ be a complex analytic function on

$$\mathbf{C} = \{x + iy \mid x, y \in \mathbf{R}\}.$$

Explain why the level curves of u and v meet orthogonally. Namely, explain why at each point (a,b) the tangent vectors to the curves $\{(x,y) \mid u(x,y) = u(a,b)\}$ and $\{(x,y) \mid v(x,y) = v(a,b)\}$, respectively, are orthogonal to each other.

b) Does there exist a complex analytic function f on $\mathbf{C}$ that satisfies the following?

$$\operatorname{Re} f(z) = \sin x, \qquad \text{for } z = x + iy.$$

Justify your answer.

c) Suppose a complex analytic function $f : \mathbf{C} \to \mathbf{C}$ on the whole complex plane $\mathbf{C}$ satisfies $|f(z)| \leq |z|^{1/2}$ for each $z \in \mathbf{C}$. Assume $f(1) = 1$. Show that such f does not exist.

5. Let D be the open unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$ in the complex plane $\mathbf{C}$.

a) Prove the following special case of the Schwarz's lemma, using the maximum principle (which is also called the maximum modulus principle). (You are *not* allowed to use Schwarz's lemma).

Let $f : D \to \mathbf{C}$ be a complex analytic function, with $f(0) = 0$ and $|f(z)| \leq 1$ on D. Then $|f(z)| \leq |z|$ for all $z \in D$.

(Hint: Apply the maximum principle to the domain $D_R = \{|z| \leq R\}$ with $0 < R < 1$.)

b) Let $f : D \to \mathbf{C}$ be a compelx analytic function, with $f(0) = 1$. Suppose $\operatorname{Re} f(z) > 0$. Prove one of the inequalities in

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} \qquad \text{for every } z \in D.$$

(You will get *full* credit by showing only one of these. You can use (a).)

6. Evaluate the following integral, using contour integration, carefully justifying each step:

$$\int_0^\infty \frac{\cos x}{(1 + x^2)^2} dx$$

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Part II: Linear Algebra and Differential Equations (Applied Exam)

1. Consider the following linear system

$$\begin{aligned} x + 3y - 2z + 2w &= 1 \\ y + z - 2w &= 2 \\ x + 2y - 2z + aw &= 0 \\ 2x + 8y - z + w &= b \end{aligned}$$

For which values of a and b, if any, does the system have: (Justify your answers!)

i) No solution? ii) Exactly one solution?

iii) Exactly two solutions? iv) More than two solutions?

2. In parts a) – c) below, we let V be a finite dimensional vector space over $\mathbf{R}$. A linear map $f : V \to V$ is called an involution of V if $f(f(x)) = x$ holds for all $x \in V$.

   a) Which eigenvalues can occur for an involution?

   b) Assume f and g are involutions of V. Show that $f \circ g$ is an involution if and only if $f \circ g = g \circ f$ holds.

   c) Assume f is an involution of V. Show that f is diagonalizable.

3. a) Let V be a finite dimensional vector space over the real numbers $\mathbf{R}$. Suppose that $v_1, \dots, v_n$ and $w_1, \dots w_m$ are both linearly independent sets which span V. Then show that $n = m$. (Do *not* quote the theorem that the cardinality of a basis is independent of the chosen basis; the problem is asking you to prove that assertion!)

   In parts b), c) d) below, we let A denote a 7×7 real matrix with characteristic polynomial

$$P_A(\lambda) = (\lambda^2 + 2\lambda + 7) \cdot (\lambda - 2)^3 \cdot (\lambda - 1) \cdot (\lambda - 3).$$

   For such A, we let

$$B = (A - 2I)^3 (A - I)(A - 3I),$$

   where I is the 7×7 identity matrix.

   b) Calculate the rank of the matrix B.

   c) Let $W \subset \mathbf{R}^7$ denote the image of the linear transformation of $\mathbf{R}^7$ given by $v \mapsto Bv$. Show that if $w \in W$, then $Aw \in W$, so that $T : w \mapsto Aw$ is a linear transformation of the vector space W.

   d) With T and W as in part c), calculate the characteristic polynomial of T (as a transformation of W).

4. Consider the following system of differential equations:

$$\frac{du}{dt} = 3u - 4v$$
$$\frac{dv}{dt} = u - v.$$

   (a) Find the solutions $u(t), v(t)$.

   (b) Is it possible to specify initial values u_0, v_0 for $t = 0$, such that the trajectory $(u(t), v(t))$ remains in one of the four quadrants of the (u, v)-plane for $-\infty < t < \infty$? Justify your answer.

5. Bernoulli's differential equation

$$\frac{dy}{dx} = f(x)y + g(x)y^a \qquad (a \neq 1)$$

can be solved using the substitution $u = y^{1-a}$.

   a) Determine the differential equation in u.

   b) Solve the differential equation

$$x\frac{dy}{dx} - y^2 \log x + y = 0$$

   for the initial condition $y(1) = c$.

   c) Determine the range of c such that $y(x)$ is defined for $0 < x < \infty$.

6. Consider the partial differential equation

$$x\, u_x + (x + y)\, u_y = 1$$

which satisfies $u(1, y) = y$ for $y \in [0, 1]$.

   a) Find the solution $u(x, y)$ using the method of characteristics.

   b) Determine the region $\{x \geq 0, y \geq 0\}$ where u is uniquely determined.