Differential Equations Qualifying Exam

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1. Consider the matrix

$$A = \frac{1}{3} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix} .$$

   (a) Determine the eigenvalues of A.

   (b) Find eigenvectors correspoding to each of the eigenvalues from part (a).

   (c) Determine

$$\lim_{n \to \infty} A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

2. Let V be a finite-dimensional inner product space, and let $W_1, \ldots, W_k$ be subspaces of V. Let W be the subspace generated (spanned) by $W_1, \ldots, W_k$. Show that $W^\perp = \bigcap_{i=1}^{k} W_i^\perp$. Here $W^\perp$ denotes the orthogonal complement to W in V with respect to the inner product.

3. For $n \geq 1$ let $V_n = \mathbb{R}[x]^{<n}$ be the vector space of real polynomials of degree less than n. For an integer k define a linear functional $\varphi_k \in V_n^*$ by $\varphi_k(f) = f(k)$.

   (a) Show that $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for the dual space V_n^*.

   (b) Let $D \colon V_3 \to V_3$ be the differentiation map (so that $D(x^2 + 2x + 1) = 2x + 2$). Find the matrix of the adjoint $D^* \colon V_3^* \to V_3^*$ in the basis from part (a).

4. The homogeneous differential equation

$$t^2 y'' - 2ty' + 2y = 0 ,$$

   where $y = y(t)$ is defined over the open interval $0 < t < 2$, has a non-trivial solution $y_1 = t^2$.

   (a) Use reduction of order to find a second solution y_2.

   (b) Show that y_1 and y_2 form a fundamental set of solutions.

(c) Find the particular solution that satisfies the initial conditions $y(1) = 3$ and $y'(1) = 4$.

5. The following nonlinear differential equations form an autonomous system:

$$\frac{dx}{dt} = 2y + xy^2,$$
$$\frac{dy}{dt} = 8x - \frac{y^3}{3}.$$

(a) Determine the critical points of this autonomous system.

(b) Determine the type and stability of the critical points.

(c) Find implicit functions $H(x, y) = c$ on which the trajectories of the system lie.

6. A test tube of length L is initially filled with pure water and sealed. At time $t = 0$, the seal is removed and the water surface comes into contact with ambient oxygen that is soluble in water. Henry's law dictates that the concentration of oxygen just under the water surface is fixed at a constant value C_s for all times. The subsequent diffusion of oxygen into the water, at a diffusivity D, is governed by the one-dimensional heat equation.

(a) Set up the partial differential equation and initial and boundary conditions for solving for the oxygen concentration $C(x, t)$ in the water, where x is the spatial coordinate pointing from the bottom of the test tube upward, and t is time.

(b) Describe the oxygen distribution in the water at the limit $t \to \infty$.

(c) Find a series solution for $C(x, t)$.