Math 101—Final Examination

April 18, 2016

Duration: 150 minutes

Surname (Last Name)   Given Name   Signature

Student Number   Section

Do not open this test until instructed to do so! This exam should have 12 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	8		7	7	
2	8		8	7	
3	6		9	7	
4	6		10	7	
5	6		11	7	
6	6		**Total**	75	

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** Calculate $\displaystyle\int_0^\pi x\cos x\,dx$. Simplify your answer completely. Answer: $\boxed{}$

1b. **[2 pts]** Which integral represents the area between the graphs of $y = 2^{-x}$ and $y = 1 - x$?

Answer: $\boxed{}$

A: $\displaystyle\int_0^1 \big(2^{-x} - (1-x)\big)\,dx$ **C**: $\displaystyle\int_0^1 \big((2^{-x})^2 - (1-x)^2\big)\,dx$ **F**: $\displaystyle\int_0^1 \big((1-x) - 2^{-x}\big)\,dx$

B: $\displaystyle\int_{-1}^0 \big(2^{-x} - (1-x)\big)\,dx$ **D**: $\displaystyle\int_{-1}^0 \big((1-x)^2 - (2^{-x})^2\big)\,dx$ **H**: $\displaystyle\int_{-1}^0 \big((1-x) - 2^{-x}\big)\,dx$

1c. **[2 pts]** Which integral equals $\displaystyle\int_0^{\pi/2} f(\sin x)\,dx$? Answer: $\boxed{}$

J: $\displaystyle\int_0^1 f(u)\,du$ **M**: $\displaystyle\int_0^{\arcsin(\pi/2)} f(u)\,du$ **Q**: $\displaystyle\int_0^{\pi/2} f(u)\,du$

K: $\displaystyle\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\,du$ **N**: $\displaystyle\int_0^{\arcsin(\pi/2)} \frac{f(u)}{\sqrt{1-u^2}}\,du$ **R**: $\displaystyle\int_0^{\pi/2} \frac{f(u)}{\sqrt{1-u^2}}\,du$

L: $\displaystyle\int_0^1 f(u)\cos u\,du$ **P**: $\displaystyle\int_0^{\arcsin(\pi/2)} f(u)\cos u\,du$ **S**: $\displaystyle\int_0^{\pi/2} f(u)\cos u\,du$

1d. **[2 pts]** Which fraction equals the decimal $0.\overline{321} = 0.321321321\ldots$? Answer: $\boxed{}$

T: $1/3$ **X**: $102/333$

V: $322/999$ **Y**: $8/27$

W: $321/1000$ **Z**: $107/333$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[2 pts]** Only one of the following statements is always true; determine which one is true.

(Assume $f(x)$ and $g(x)$ are continuous functions.) Answer: $\boxed{}$

A: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

B: If $f(x)$ is an odd function, then $\int_{-3}^{-2} f(x)\,dx = \int_{2}^{3} f(x)\,dx$.

C: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

D: If $g(x) \geq f(x) \geq 0$ for all x and $\int_1^{\infty} g(x)\,dx$ converges, then $\int_1^{\infty} f(x)\,dx$ converges.

E: $\displaystyle\int_0^7 f(x^2)\,x\,dx = \int_0^7 f(u)\,\frac{du}{2}$.

2b. **[2 pts]** Which integral equals $\displaystyle\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right)$? Answer: $\boxed{}$

F: $\displaystyle\int_0^1 e^{-1-2x}\,dx$

J: $\displaystyle\int_0^2 e^{-1-2x}\,dx$

G: $\displaystyle 2\int_0^2 e^{-1-2x}\,dx$

K: $\displaystyle\int_1^3 e^{-x}\,dx$

H: $\displaystyle\int_0^2 e^{-x}\,dx$

2c. **[2 pts]** For which values of the constant q does the sum $\displaystyle\sum_{n=1}^{\infty}\frac{(\log n)^q}{n}$ converge?

L: It converges only when $q < -1$.
M: It converges only when $q \leq -1$.
N: It converges only when $q > -1$.
P: It converges only when $q \geq -1$.
Q: It converges for all q.

Answer: $\boxed{}$

2d. **[2 pts]** Calculate $\displaystyle\int_{-2}^{2} xe^{x^2}\,dx$. Simplify your answer completely. Answer: $\boxed{}$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** If $F(x)$ is defined by $F(x) = \displaystyle\int_{x^4-x^3}^{x} e^{\sin t}\,dt$, find $F'(x)$.

3b. **[3 pts]** Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$. Your answer cannot contain any inverse trigonometric functions.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Find the function $y = y(x)$ that satisfies $y(1) = 4$ and

$$\frac{dy}{dx} = \frac{15x^2 + 4x + 3}{y}.$$

4b. **[3 pts]** Calculate $\displaystyle\int \frac{12x + 4}{(x-3)(x^2+1)}\, dx.$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge? Justify your conclusion.

5b. **[3 pts]** Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge? Justify your conclusion.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Find the sum of the convergent series $\displaystyle\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$. Simplify your answer completely.

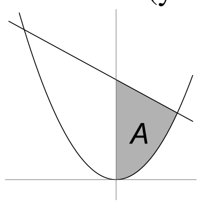
6b. **[3 pts]** Determine the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$.

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. **[7 pts]** Find the average value of the function $f(x) = 3\cos^3 x + 2\cos^2 x$ on the interval $0 \le x \le \frac{\pi}{2}$. Simplify your answer completely.

8. Let A be the region to the right of the y-axis that is bounded by the graphs of $y = x^2$ and $y = 6 - x$. Both parts of this question concern this region A.

(a) **[4 pts]** Find the centroid of A, assuming it has constant density $\rho = 1$. The area of A is $\frac{22}{3}$ (you don't have to show this). You may leave your answer in calculator-ready form.



(b) **[3 pts]** Write down an expression, using horizontal slices (disks), for the volume obtained when the region A is rotated around the y-axis. Do not evaluate any integrals; simply write down an expression for the volume.

9. Both parts of this question concern the integral $I = \int_0^2 (x-3)^5\,dx$.

(a) **[3 pts]** Write down the Simpson's Rule approximation to I with $n = 6$. You may leave your answer in calculator-ready form.

(b) **[4 pts]** Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? Justify your answer. You may use the formulas

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4},$$

where K is an upper bound for $|f''(x)|$ and L is an upper bound for $|f^{(4)}(x)|$.

10. Define $f(x) = \int_0^x \frac{1 - e^{-t}}{t}\, dt$.

(a) **[4 pts]** Show that the Maclaurin series for $f(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

(b) **[3 pts]** Use the Ratio Test to answer the question: for which values of x does the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$ converge?

11. Both parts of this question concern the series $S = \sum_{n=1}^{\infty} (-1)^{n-1} 24 n^2 e^{-n^3}$.

(a) **[4 pts]** Show that the series S converges absolutely.

(b) **[3 pts]** Suppose that you approximate the series S by its fifth partial sum S_5. Give an upper bound for the error resulting from this approximation. Explain why your error bound is valid for this series.