Marks

1. Multiple Choice Questions: Select ONE correct answer (a, b, c, d, or e) for each question and write it in the table at the bottom of the page. You will not be graded for any work or answers outside those boxes.

(Q1) The value of the integral $\int_0^1 \mathrm{e}^{(x+\mathrm{e}^x)}\,dx$ is

(a) $\dfrac{\mathrm{e}^{(x+\mathrm{e}^x+1)}}{(x+\mathrm{e}^x+1)}$ **(b)** $\mathrm{e}^{\mathrm{e}}-\mathrm{e}$ **(c)** $\mathrm{e}^{\mathrm{e}}+\mathrm{e}$ **(d)** $\mathrm{e}^{\mathrm{e}+1}$ **(e)** $(1-\mathrm{e})\mathrm{e}$

(Q2) An antiderivative of $\dfrac{x}{4-x^2}dx$ is

(a) $\dfrac{1}{4}\arctan(x/2)$ **(b)** $\dfrac{1}{2}x\arctan(x/2)$ **(c)** $\dfrac{1}{2}\ln\dfrac{|2+x|}{|2-x|}$ **(d)** $-\dfrac{1}{2}\ln|4-x^2|$ **(e)** $\dfrac{1}{2}\ln|4-x^2|$

(Q3) An approximation to the integral $\int_0^2 \arctan(x)\,dx$ using n rectangles would be

(a) $\sum_{k=0}^{2}\arctan(k)$ **(b)** $\sum_{k=1}^{n}\arctan(\frac{k}{n})$ **(c)** $\frac{2}{n}\sum_{k=1}^{n}\arctan(\frac{k}{n})$ **(d)** $\frac{1}{n}\sum_{k=0}^{2}\arctan(\frac{k}{n})$ **(e)** $\frac{2}{n}\sum_{k=1}^{n}\arctan(\frac{2k}{n})$

(Q4) The value of the integral $\int_1^{\mathrm{e}} \dfrac{\ln(x)}{x^2}\,dx$ is

(a) $1-\dfrac{2}{\mathrm{e}}$ **(b)** $1+\dfrac{2}{\mathrm{e}}$ **(c)** 1 **(d)** $\dfrac{1}{\mathrm{e}^2}-1$ **(e)** $\dfrac{\ln(\mathrm{e})}{\mathrm{e}^2}-\ln(1)$

(Q5) Consider a mass density function $d(x)=4-x^2$ over the interval $0\le x\le 2$. The centre of mass of this mass distribution is at

(a) 1 **(b)** 4 **(c)** $\dfrac{4}{3}$ **(d)** $\dfrac{3}{4}$ **(e)** $\dfrac{16}{3}$

Q1	Q2	Q3	Q4	Q5

NOTE: carefully check to ensure that you have correctly matched the response with the relevant questions. Only answers in this table using letters a, b, c, d, or e will be graded for Problem 1. Illegible or ambiguous responses will not receive marks.

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[10] **2.** A certain random variable X takes values in the interval $0 \leq x \leq 2$, with probability density $p(x) = 1 - x/2$. Find the mean and variance of X.

Mean = $\boxed{}$

Variance = $\boxed{}$

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[12] **3.** Consider the cumulative distribution function, $F(x)$ on $0 \leq x \leq 10$ shown in Figure 1.

(a) Use the grid in Figure 2 to sketch the probability density $p(x)$.

(b) Indicate the **median** of this distribution on the appropriate figure.

(c) On both figures, indicate the probability that the value of x is in the interval $5 \leq x \leq 7$

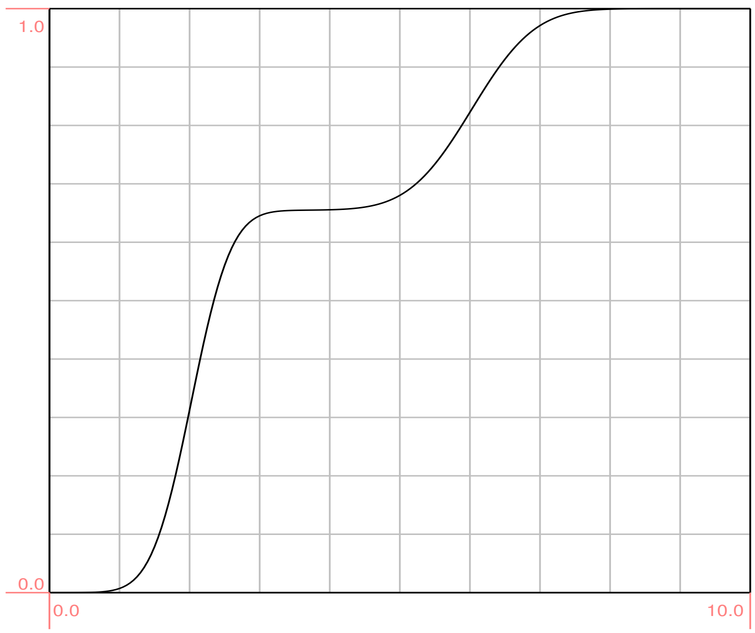

Figure 1

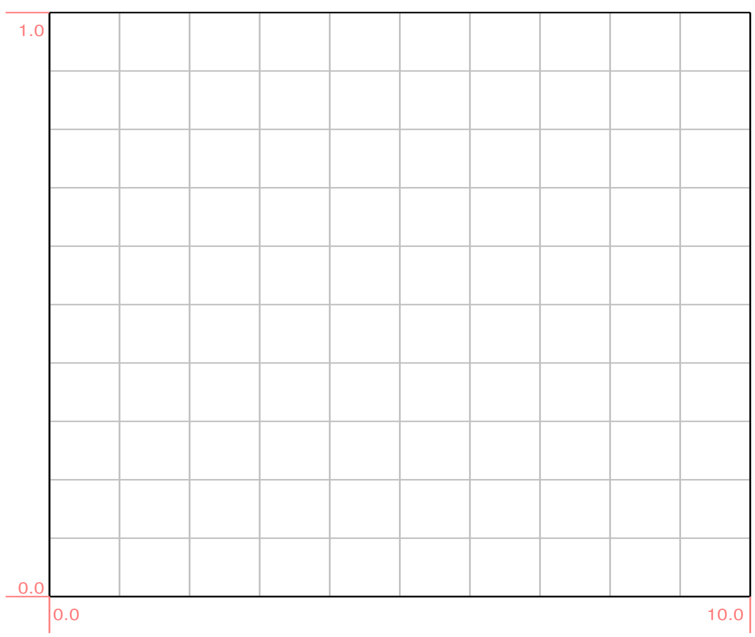

Figure 2

Continued on page 5

[10] **4.** The level of water in a reservoir changes over the year due to inflow and outflow. If the volume at $t = 0$ is 10^4 cubic metres and the rate of change of volume is $10\sin(\pi t/10)$ cubic metres per day, where t is the time in days, find the total change in the volume between $t = 0$ and $t = 5$.

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[12] **5.** Consider the curve given by

$$y = f(x) = \frac{1}{x^p}, \quad 1 \leq x \leq B$$

where p, B are constants, with $B > 1$ and $p > 0$.

(a) Compute the area of the region between this curve and the x axis for $1 \leq x \leq B$. (Later, we will refer to this area as A.)

$A = \boxed{}$

(b) Compute the volume obtained by revolving the same region about the x axis. (Later, we will refer to this volume as V.)

$V = \boxed{}$

(c) Now consider the limit as $B \to \infty$. For what range of values of the constant p is it true that the volume V has a finite limit while the area A becomes infinite? Explain your answer in terms of your calculations above (1 sentence).

Range $= \boxed{}$

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[12] **6.** A CT scanner in a hospital is switched on every morning and switched off at night. The scanner has a probability $q = 0.001$ of failure when the technician turns it on in the morning. (Assume that if it turns on, it will not fail later that day, i.e. treat this as a **discrete**, event. Also, assume that the failure is independent of age or history of the machine.)

(a) What is the probability that the machine will work on days 1 to $k-1$ and then fail on day k?

Probability = $\boxed{}$

(b) Use your result in (a) to find the mean time to failure of this machine, i.e. the expected value of the number of the day on which it first fails. You may find it helpful to use the summation formula

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2} \qquad \text{for } |r| < 1$$

Mean time to failure = $\boxed{}$

(c) Use the properties of an infinite geometric series to obtain the summation formula given in (b).

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[12] **7.** Cholera is a disease that can spread through untreated sewage polluting the drinking water supply. Let $y(t)$ be the fraction of people in a population who have the disease at time t. Assume that in a certain city, the fraction of people who have the disease increases at a rate proportional to the fraction of people who do not yet have the disease. (We will use $k > 0$ as the constant of proportionality).

(a) Write down the differential equation that $y(t)$ satisfies.

DE: $\boxed{}$

(b) Solve this differential equation, assuming that at time $t = 0$ the fraction of people who have the disease is $y(0) = y_0$, where $0 \leq y_0 \leq 1$.

$y(t) = \boxed{}$

(c) Use one clear sentence and a simple sketch to explain what happens to the fraction of infected individuals as time goes by.

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[12] **8.** Taylor Series.

(a) Write down the first five terms in a Taylor series expansion of the function $y = f(x) = \mathrm{e}^x$ about $x = 0$. (Note: If you remember this series, you are not asked to "derive" it.)

$f(x) =$ $\boxed{}$

(b) Use the **first three terms** of this series to find a numerical estimate for the constant e (the base of natural logarithms).

$\mathrm{e} \approx$ $\boxed{}$

(c) Use the **first three terms** of this series to find a numerical approximation of the integral shown below:

$$\int_0^1 \mathrm{e}^{-x^2}\, dx$$

Leave the answer in terms of fractions rather than decimals.

$\int_0^1 \mathrm{e}^{-x^2}\, dx \approx$ $\boxed{}$

Continued on page 10

Some useful formulae

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan^2(x) + 1 = \sec^2(x)$$

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The University of British Columbia

Sessional Examinations - April 2005

Mathematics 103

Integral Calculus with applications to Life Sciences

R. Israel: sections 201 (morning) and 202 (afternoon)
L. Keshet: sections 203 (morning) and 204 (afternoon)

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________________ **Signature** ______________________________

Student Number________________________ **Section**__________________________

Special Instructions:

No calculators, books, notes, electronic devices or other aids. Unless otherwise indicated, show all your work. Answers not supported by calculations or reasoning may not receive credit. Excessively messy work will not be graded. Write your answers in the boxes where these are provided. The last page contains some helpful formulae, and blank space that can be used for rough work that you don't want the markers to look at (e.g. for question 1).

Rules governing examinations

1		20
2		10
3		12
4		10
5		12
6		12
7		12
8		12
Total		100