Mathematics 104-184 Final Examination

Final Examination — December 14 Duration: 2.5 hours

*This test has **13 questions** on **18 pages**, for a total of 100 points.*

- Q1-Q8 are short-answer questions worth 48 pts [3 pts each]; put your answer in the boxes provided.
- Q9-Q13 are long-answer and worth 52 pts; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Use the four blank pages is you need extra space, but you must leave a clear note that this has been done.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Please circle your course and section:

MATH 104	101	102	103	104	105	106	108	109
MATH 184	101	102	103	104	105	106	I don't know.	

- PRINT your name and ID # very clearly. Failure to do so may result in a grade of 0:

First Name: _____ Last Name: _____

Student-No: _____ Signature: _____

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

3 marks 1. (a) Evaluate $\lim_{x \to -7} \dfrac{2x^2 + 11x - 21}{x^2 + 8x + 7}$.

Answer:

3 marks (b) How long would it take an investment of \$100,000 with an interest rate of 6% compounded continuously to gain \$8000?

Answer:

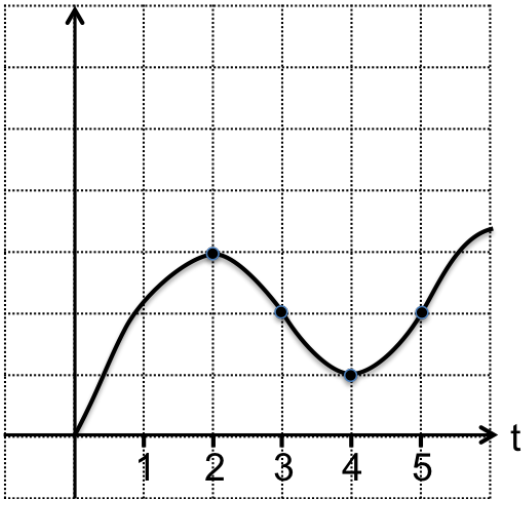
3 marks

2. (a) A ball is thrown upward from a roof of a 50 foot building with initial velocity of 96 ft/sec. The height after t seconds is given by $h(t) = 50 + 96t - 16t^2$. Determine the maximum height that the ball reaches. Simplify your answer.

Answer:

3 marks

(b) The graph of the position of a particle is shown below, where t is measured in second and the dots are local extrema or points of inflection. Determine when the particle is speeding up. Hint: a particle is "speeding up" when its velocity and acceleration have the same sign.



Answer:

3 marks 3. (a) Find the derivative of $y = e^{\sin(x^2)}$.

Answer:

3 marks (b) Compute the Taylor polynomial of degree 3 of $f(x) = x \ln x$ at $a = 1$.

Answer:

3 marks 4. (a) Use linear apporimation to estimate $\sqrt{69}$.

Answer:

3 marks (b) Decide whether the estimate from 5(a) is an over or under estimate and then use the formula, $\frac{M}{2}|x-a|^2$, to determine the worst case error.

Answer:

3 marks 5. (a) Find the absolute minimum $y = x\sqrt[3]{x-3} = x(x-3)^{1/3}$ on the interval $[0, 4]$

Answer:

3 marks (b) Let $f(x) = x + e^x$. Find the equation of the tangent line of $y = f^{-1}(x)$ at $x = 1$. Recall that $f^{-1}(x)$ is the inverse function of $f(x)$ and its derivative can be calculated using formula $\dfrac{\mathrm{d}f^{-1}}{\mathrm{d}x} = \dfrac{1}{f'(f^{-1}(x))}$.

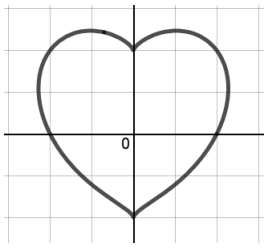
Answer:

3 marks 6. (a) Suppose that a company's Cost function is an increasing and differentiable function. Use *calculus* to show that average cost will be minimized at the level of production when the average cost is equal to the marginal cost.

3 marks (b) A company's demand curve is given by $p+10q = 168$. Determine the selling price which would produce the most revenue.

Answer:

3 marks 7. (a) Consider the equation $(x^2 + y^2 - 1)^3 = x^2y^3$ with the following graph:



Find the slope of the tangent line at the point (1, 1).

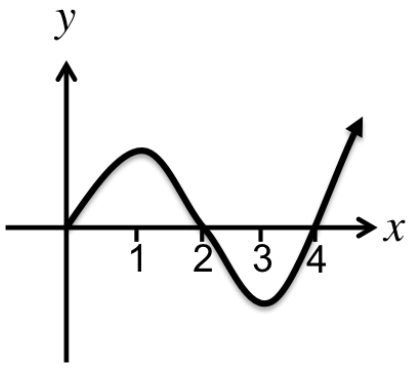
Answer:

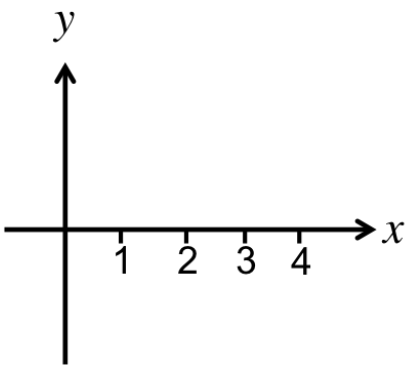
3 marks (b) Interpret the following limit as the slope of a tangent line to some curve $y = f(x)$ at a point $x = a$. Determine what $f(x)$ and a are and then use differentiation rules to evaluate the limit:

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

Answer:

3 marks 8. (a) The graph on the left is the derivative of $y = f'(x)$. Given that $f(0) = 0$, sketch the graph of $f(x)$ on the right grid. Clearly lable any local extrema and inflection points.





3 marks (b) Consider the following function:

$$f(x) = \begin{cases} 4 - x^2, & x < 1 \\ \ln x, & 1 < x < 3 \\ e^x, & x > 3. \end{cases}$$

Evalulate the limit

$$\lim_{x \to 1^-} f(f(x)).$$

Answer:

Full-solution problems - 10 pts each, excpet #11 which is worth 12pt: Justify your answers and **show all your work** for problems 9 and 13. Place a box around your final answer. Unless otherwise indicated, **simplification of answers is required in these questions to earn full credit**.

9. Consider the following function:

$$f(x) = \begin{cases} e^x, & x \leq 0 \\ ax + b, & 0 < x \leq 1 \\ 1 + \sqrt{x} & x > 1. \end{cases}$$

4 marks (a) Determine values for a and b for which $f(x)$ is continuous.

Answer:

6 marks (b) Determine whether or not $f(x)$ can be differentiable at $x = 0$ or $x = 1$.

Answer:

10 marks 10. A company's demand and cost curves are given by:

$$p + \sqrt{q} = 150 \quad \text{and} \quad C(q) = 2500 + 6q.$$

Determine the selling price which would produce the most profit. To receive full credit you must justify your solution and simplify your answer.

Answer:

11. Consider the function $f(x) = \dfrac{x^2 - 2}{(x-2)^2}$, and its derivatives

$$f'(x) = \frac{4(1-x)}{(x-2)^3}, \qquad f''(x) = \frac{4(2x+1)}{(x-2)^4}.$$

3 marks (a) Find all the asymptotes of $f(x)$

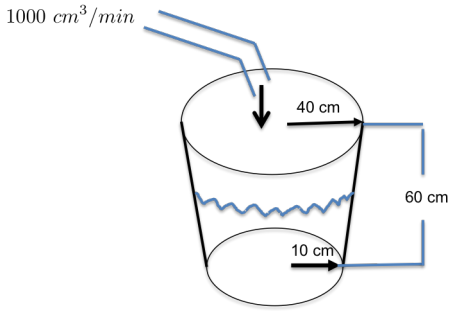
3 marks (b) Find the intervals on which $f(x)$ is increasing or decreasing and classify the local extreme values.

3 marks (c) Determine where $f(x)$ is concave up or down and find the inflection points.

3 marks (d) Sketch the graph of $y = f(x)$.

10 marks 12. Suppose you are given a 12" x 12" square piece of cardboard and asked to construct a box by cutting out squares of equal size from the four corners and bending up the sides. Find the dimensions of the resulting box that has the largest volume. Explain carefully how you may conclude that your answer is guaranteed to be a box of maximal volume.

10 marks 13. A bucket is 60 cm high, has a radius of 40 cm at the top and 10 cm at the bottom. Water is being dumped into the bucket at a rate of 1 L/min. How fast is the water level rising when the water is 30 cm deep? You may use the fact that the volume of a trunctated cone with height h, radius at the bottom r_1 and radius at the top r_2, is given by $V = \frac{1}{3}\pi h(r_2^2 + r_1 r_2 + r_1^2)$.



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