Mathematics 105
April 2005

1. (10 marks) Let $f(x) = \dfrac{4}{5 - 2x}$. Find the average value of $f(x)$ over the interval $0 \le x \le 2$.

2. (10 marks) The demand q for a product is related to the unit price p by the demand equation

$$p = D(q) = 10(1 + 4q)^{-1/2}.$$

Find the consumer surplus if $p = 2$. Please simplify. Note that if $p = 2$ then $q = 6$.

3. (10 marks) Let $\mathcal{A}$ be the part of the *first* quadrant that lies on or below the line $y = 3$ and on or above the curve $y = \sqrt{1 + x^3}$. Use the trapezoidal rule with $n = 2$ to estimate the area of $\mathcal{A}$. A calculator-ready answer is enough.

4. (5 + 5 marks) The random variable X has probability density function $f(x)$, where

$$f(x) = \begin{cases} 4x - 4x^3 & \text{if } 0 \le x \le 1; \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the probability that X is less than or equal to $1/2$. Please simplify.

(b) Find the mean (expected value) of X. Please simplify.

5. (8 + 2 marks) The Museum of Calculus is currently 50 meters from the edge of an eroding cliff. Geologists predict that at time t, the distance y, in meters, from the Museum to the cliff edge will be changing at a rate given by

$$\frac{dy}{dt} = \frac{-120e^{-0.5t}}{(1 + 4e^{-0.5t})^2}.$$

Here t is measured in years, and $t = 0$ is now.

(a) Find a formula for y in terms of t.

(b) Will the distance from the Museum to the cliff edge ever reach 0? Explain.

6. (8 + 2 marks) Suppose that $y(0) = 1/5$ and

$$\frac{dy}{dt} = y^2 e^{-t}$$

for all t.

(a) Find an explicit formula for y as a function of t.

(b) Find $\lim\limits_{t \to \infty} y(t)$.

7. $(3+7$ marks) The HigherPrices corporation (HP) has a local monopoly on inkjet printers and the proprietary ink cartridges for them.

The selling price $f(x,y)$, in dollars, of a printer is connected to the number x of printers and the number y of cartridges sold per week by the equation $f(x,y) = 300 - x$. Note there is no dependence on y — shoppers do not think about the cost of cartridges when buying a printer. The cost to HP of a printer is $200.

The selling price $g(x,y)$, in dollars, of a cartridge is connected to the number x of printers and the number y of cartridges sold per week by the equation $g(x,y) = 61 - \dfrac{3y}{x}$. The cost to HP of a cartridge is $1.

(a) Let $P(x,y)$ be HP's weekly profit if it sells x printers and y cartridges. Write down an explicit expression for $P(x,y)$.

(b) How many printers and how many cartridges should HP sell to maximize its weekly profit? You need not give a justification that the numbers calculated produce the *maximum* profit.

8. (10 marks) By employing x semi-skilled workers and y skilled workers, a factory can assemble $(4xy + y^2)^{1/2}$ custom-built computers per hour. The factory pays each semi-skilled worker $8 per hour, and each skilled worker $20 per hour. Use the method of Lagrange multipliers (no credit will be given for any other method) to determine the maximum number of computers the factory can assemble in an hour for a total labour cost of $720. Please simplify. You need not show that the answer you compute is actually the *maximum*.

9. $(2+8$ marks) If $b^2 - 4c > 0$, then the equation $x^2 + bx + c = 0$ has two real unequal roots. Let $f(b,c)$ be the larger of these roots.

(a) Write down a formula for $f(b,c)$.

(b) It is easy to verify that -1 is the larger of the two roots of $x^2 + 5x + 4 = 0$. Use a suitable linear approximation to $f(b,c)$ to estimate the larger of the two roots of the equation $x^2 + 5.03x + 4.06 = 0$. (No credit will be given for a method that does not use linear approximation.) Please simplify.

10. (10 marks) Taxation law allows a firm to claim a tax credit for the depreciation of an asset. In *straight line depreciation*, if the asset is worth A at time $t = 0$, and is depreciated to value 0 at time T, then the tax credit flows in according to a continuous income stream. The rate of flow $f(t)$ at time t is given by $f(t) = 2A(T - t)/T^2$ for $0 \le t \le T$. This means that the tax credit generated between t and $t + dt$ is approximately $f(t)\,dt$.

Let t be measured in years, let $t = 0$ be the present, and suppose that your company owns an oil well now worth 100 (million dollars). So $A = 100$. Suppose your company is allowed to depreciate the well to value 0 in 10 years, using straight line depreciation. So $T = 10$. Suppose also that the prevailing interest rate is 5%, compounded continuously. Find the present value of the depreciation tax credits for the oil well.