The University of British Columbia

Final Examination - April 13, 2011

Mathematics 152

All Sections

Closed book examination. No calculators.   Time: 2.5 hours

Last Name _____ **First** _____ **Signature** _____

Student Number _____

Section :   _____

Instructor :   _____

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

For questions **A1-A3** below, **x** and **y** are the vectors given below:

$$\mathbf{x} = [0, -3, 2]$$
$$\mathbf{y} = [1, 1, 1]$$

A1: Compute $\mathbf{x} + \mathbf{y}$.

A2: Compute $\mathbf{x} \times \mathbf{y}$ (the cross product of **x** and **y**).

A3: Compute the projection of **x** onto the direction of **y**.

A4: Circle the *one* correct answer below. A linear system of two equations in three unknowns has

(**a**) always a unique solution.

(**b**) either a unique solution or no solutions.

(**c**) either a unique solution or an infinite number of solutions.

(**d**) either no solutions or an infinite number of solutions.

(**e**) either a unique solution or exactly two distinct solutions.

A5: Find the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

A6: What is output after the following lines of MATLAB code?

```
A = [1 2 3; 4 5 6; 7 8 9];
A(2,:)
```

A7: Write the matrix `A` that would result from the following lines of MATLAB code:

```
A = zeros(3,2);
A(1,1) = 5;
A(2,2) = 3;
A(3,1) = 1;
```

A8: Write the matrix `A` that would result from the following lines of MATLAB code:

```
A = zeros(4,4);
for k=1:4
    A(k,k) = -k;
end
for k=1:4
    A(1,k) = 1;
end
```

For questions **A9-A11** below consider the circuit in the diagram below:
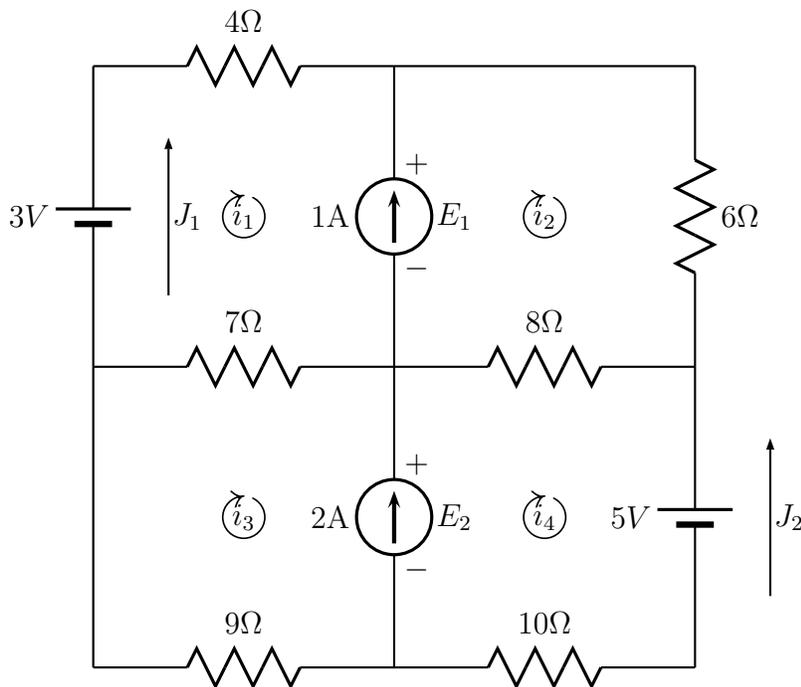
A9: List the unknowns in the linear system for this circuit using the technique of loop currents you learned in the computer labs this term.

A10: In terms of these unknowns, write the linear equation that represents Kirchhoff's law of voltage drops around the third loop (corresponding to i_3) in the circuit above.

A11: Write a linear equation that expresses the current through the current source 1 in terms of the loop currents in the diagram.

A12. Write the matrix of the rotation which transforms $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

The unit square in $\mathbb{R}^2$ is the set of all points (x, y) with $0 \le x \le 1$ and $0 \le y \le 1$. In questions **A13** and **A14** below, suppose each point in the unit square undergoes a linear transformation that has the given matrix representation A. Draw a labelled sketch of the transformed points.

A13: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A14: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

A15: Consider the linear transformation T which takes vectors in $\mathbb{R}^2$, first rotates them by $\pi/6$ (30°) counter-clockwise and then projects the result onto the direction $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$. *Note:* $\cos \pi/6 = \sqrt{3}/2$ and $\sin \pi/6 = 1/2$. Your final answer should not contain trigonometric functions.

A16: Find a vector of length 1 that is parallel to the line given by the equations

$$\begin{aligned} y - z &= 1 \\ 2x + y + z &= 2 \end{aligned}$$

For questions **A17-A19** below, u and z are the complex numbers given below:

$$u = 4 - 2i$$
$$z = -1 + 3i$$

A17: Compute uz. Your answer should be in the form $a+ib$ where a and b are real numbers.

A18: Compute $|z|$.

A19: Evaluate u/z. Put your answer in the form $a + ib$ where a and b are real numbers.

A20: Write $-1 + i$ in polar form $re^{i\theta}$.

A21: Suppose that A is a 12×12 invertible matrix and b is a column vector with 12 entries. Both A and b have been entered into MATLAB. What commands would you use to compute the vector x that solves the linear system

$$A\mathbf{x} = \mathbf{b}?$$

Questions **A22-A24** below concern a random walk with three states that has transition matrix

$$P = \begin{bmatrix} 8/10 & 3/10 & 1/10 \\ 2/10 & 3/10 & 7/10 \\ 0 & 4/10 & 2/10 \end{bmatrix}.$$

A22: If the walker is equally likely to start in each of the three states, what is the chance that she will be in state 3 at the next time step?

A23: If the walker starts in state 2, which state will she most likely be in after 2 time steps?

A24: What is the equilibrium probability vector for the random walk?

A25: The matrix

$$A = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

contains a real parameter p. For what values of p does A have three real, distinct eigenvalues?

A26: For which values of the parameters a, b and c is the matrix

$$B = \begin{bmatrix} a & a & a \\ a & a & b \\ a & b & c \end{bmatrix}$$

invertible?

A27: Circle *all* of the following statements that are true for all $n \times n$ invertible matrices A:

    (a) A has rank n.

    (b) The reduced row echelon form of A is the identity matrix.

    (c) The homogeneous problem $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

    (d) A does not have $\lambda = 0$ as an eigenvalue.

    (e) $\det(A) \neq 0$

A28: Let

    • Rot_α be the matrix for 2D rotation counterclockwise with angle α

    • Refl_β 2D reflection through the line that makes angle β with the positive x axis

    • Proj_θ 2D projection onto the line that makes angle θ with the positive x axis.

Circle *all* the matrix products below that are the identity matrix for all α, β and θ.

    (a) $\mathrm{Refl}_\beta\ \mathrm{Refl}_\beta$

    (b) $\mathrm{Rot}_\alpha\ \mathrm{Rot}_{-\alpha}$

    (c) $\mathrm{Proj}_\theta\ \mathrm{Proj}_\theta$

    (d) $\mathrm{Proj}_\theta\ \mathrm{Proj}_{\pi/2-\theta}$

    (e) $\mathrm{Rot}_\alpha\ \mathrm{Refl}_{\alpha/2}$

For questions **A29** and **A30** below, decide if the statements are true or false. In each case, give a *brief justification* of your conclusion.

A29: For all $n \times n$ matrices A and B such that $\det(A) = \det(B) = 0$, $\det(A + B) = 0$ also.

A30: If vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are linearly independent and lie in the plane P that goes through the origin then $\mathbf{v} \times (\mathbf{v} \times \mathbf{w})$ also lies in P.

Part B - Long Answer Questions, 5 marks each

B1: Three identical fields of size one hectare are planted with different amounts of three kinds of wheat (types 1, 2 and 3). The first field is divided in three equal parts with one type of wheat planted on each part. It yields 12 tons of total harvest. The second field is divided in two equal parts with type 2 planted on one part and type 3 planted on the other. It yields 10 tons. The third field is divided in two equal parts with type 1 planted on one part and type 2 on the other. It yields 16 tons.

(a) [2 marks] Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the vector of unknowns, where x_k is the yield in tons per hectare of wheat type k. Describe the information above as a linear system in the form

$$A\mathbf{x} = \mathbf{b}$$

(write A and $\mathbf{b}$ with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. What is the yield in tons per hectare for each type of wheat?

B2: Consider the differential equation system

$$\mathbf{y}' = A\mathbf{y}$$

where A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ with corresponding eigenvectors

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{k}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(a) [2 marks] What is the general solution of the system of differential equations?

(b) [3] Find the solution that matches the initial conditions $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

B3: Let A be the 2×2 matrix which transforms $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and transforms $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

    (a) [2 marks] What are the eigenvalues and eigenvectors of A?

    (b) [1] Is A invertible? Justify briefly.

    (c) [2] What is the matrix A?

B4: Let $\mathbf{a} = [1, 2, -1]$. Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{x} \times \mathbf{a}$ for all $\mathbf{x}$ in $\mathbb{R}^3$, where $\times$ is the cross product in $\mathbb{R}^3$.

    (a) [1 mark] Show that T is a linear transformation.

    (b) [2] Find the matrix A that corresponds to T.

    (c) [1] Compute $\det A$.

    (d) [1] Find *any* nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

B5: Consider the matrix below:

$$A = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is known that $\lambda_1 = 1$ is an eigenvalue of A.

(a) [2 marks] Find an eigenvector associated with the eigenvalue $\lambda_1 = 1$.

(b) [2] Find the two other eigenvalues λ_2 and λ_3 of A.

(c) [1] Find eigenvectors associated with λ_2 and λ_3.

B6: Consider the linear systems described below. In cases (a)-(d), write a *single*, possible reduced row echelon form of the augmented matrix of the system.

(a) [1 mark] Two equations in two unknowns with unique solution $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) [1] Two equations in three unknowns with solutions

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where t is a parameter.

(c) [1] Three equations in three unknowns with no solution.

(d) [1] Three equations in two unknowns with no solution.

(e) [1] Repeat part (d) but in this case only, list *all possible* corresponding reduced row echelon forms.

The End