Math 200, Final Exam

December 2008

No calculators or notes of any kind are allowed. Time: 2.5 hours.

1. [**11**] A surface is given by

$$z = x^2 - 2xy + y^2.$$

(a) Find the equation of the tangent plane to the surface at $x = a,\ y = 2a$.
(b) For what value of a is the tangent plane parallel to the plane $x - y + z = 1$?

2. [**12**] The pressure in a solid is given by

$$P(s, r) = sr(4s^2 - r^2 - 2)$$

where s is the specific heat and r is the density. We expect to measure (s, r) to be approximately $(2, 2)$ and would like to have the most accurate value for P. There are two different ways to measure s and r. Method 1 has an error in s of ± 0.01 and an error in r of ± 0.1, while method 2 has an error of ± 0.02 for both s and r.

Should we use method 1 or method 2? Explain your reasoning carefully.

3. [**11**] $u(x, y)$ is defined as

$$u(x, y) = e^y F(xe^{-y^2})$$

for an arbitrary function $F(z)$.
(a) If $F(z) = \ln(z)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
(b) For an arbitrary $F(z)$ show that $u(x, y)$ satisfies

$$2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u.$$

4. [**12**] The air temperature $T(x, y, z)$ at a location (x, y, z) is given by:

$$T(x, y, z) = 1 + x^2 + yz.$$

(a) A bird passes through $(2, 1, 3)$ travelling towards $(4, 3, 4)$ with speed 2. At what rate does the air temperature it experiences change at this instant?

(b) If instead the bird maintains constant altitude ($z = 3$) as it passes through $(2, 1, 3)$ while also keeping at a fixed air temperature, $T = 8$, what are its two possible directions of travel?

5. [14]
(a) Find all saddle points, local minima and local maxima of the function

$$f(x,y) = x^3 + x^2 - 2xy + y^2 - x.$$

(b) Use Lagrange multipliers to find the points on the sphere $z^2 + x^2 + y^2 - 2y - 10 = 0$ closest to and furthest from the point $(1, -2, 1)$.

6. [13] Consider the integral

$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\, dx dy$$

(a) Sketch the region of integration.
(b) Evaluate I.

7. [14] Let R be the region bounded on the left by $x = 1$ and on the right by $x^2 + y^2 = 4$. The density in R is

$$\rho(x,y) = \frac{1}{\sqrt{x^2 + y^2}}$$

(a) Sketch the region R. (b) Find the mass of R.
(c) Find the centre-of-mass of R.
Note: You may use the result $\int \sec(\theta) d\theta = \ln|\sec\theta + \tan\theta|$

8. [13] Let

$$I = \iiint_T xz\, dV,$$

where T is the eighth of the sphere $x^2 + y^2 + z^2 \leq 1$ with $x, y, z \geq 0$.
(a) Sketch the volume T.
(b) Express I as a triple integral in spherical coordinates.
(c) Evaluate I by any method.