Math 215/255 Final Exam (Dec 2005)

Last Name: ______________________________ **First name**: ______________________________

Student #: ______________________________ **Signature**: ______________________________

Circle your section #:

Burggraf=101, Peterson=102, Khadra=103, Burghelea=104, Li=105

I have read and understood the instructions below:

Please sign:

Instructions:

1. No notes or books are to be used in this exam.
2. You are allowed to bring a letter-sized formula sheet and a small-screen, non-graphic, non-programmable calculator.
3. Justify every answer, and show your work. Unsupported answers will receive no credit.
4. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
5. At the end of the hour you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
6. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

Question	grade	value
1		12
2		12
3		14
4		14
5		12
6		16
7		20
Total		**100**

Question 1: **[12 marks]**

Solve each one of the following first-order initial value problems for a real-valued solution $y(t)$ in explicit form. Also, determine the domain of definition for each solution.

(a) $y' = y\sin t + 2te^{-\cos t}$, $y(0) = 1$.

(b) $1 - (1-t)yy' = 0$, $y(0) = 1$.

Solution:

(a) Rewrite the eqn in the normal form $y' + (-\sin t)y = 2te^{-\cos t}$. Thus the integrating factor is $\mu(t) = e^{\int(-\sin t)dt} = e^{\cos t}$. Thus

$$\left(e^{\cos t}y\right)' = e^{\cos t}\left(2te^{-\cos t}\right) = 2t \quad \Rightarrow \quad e^{\cos t}y = \int (2t)dt = t^2 + C \quad \Rightarrow \quad y(t) = e^{-\cos t}(t^2 + C).$$

$y(0) = 1 \;\Rightarrow\; C = e \;\Rightarrow\; y(t) = e^{-\cos t}(t^2 + e)$. It is define on $(-\infty,\ \infty)$.

(b) This eqn is not exact but is separable.

$$yy' = \frac{1}{1-t} \;\Rightarrow\; \frac{1}{2}y^2(t) = -\ln|t-1| + C \;\Rightarrow\; y^2(t) = -\ln(t-1)^2 + C \;\Rightarrow\; y(t) = \pm\sqrt{C - \ln(t-1)^2}.$$

$y(0) = 1 \;\Rightarrow\; C = 1 \quad \Rightarrow \quad y(t) = \sqrt{1 - \ln(t-1)^2}$. It is define on the interval $(1-\sqrt{e},\ 1)$.

Question 2: **[12 marks]**

Answer "True" or "False" to the statements below. **Put your answers in the boxes.** (20 points)

(a) Suppose the Wronskian of two functions $f(t)$ and $g(t)$ is $W(f,g)(t) = t(t-1)$ which is zero at $t = 0,\ 1$. Then, $f(t)$ and $g(t)$ must be linearly dependent functions.

False.

(b) The Laplace transform of the initial value problem $y''' + y'' + y' = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 3$ yields $Y(s) = (s^2 + 3s + 6)/(s^3 + s^2 + s)$.

True.

(c) The equation $y'' + \gamma y' + 9y = \cos(\omega t)$ describes a periodically forced spring-mass system. The larger the value of the damping coefficient γ the closer is the maximum resonance frequency (ω_{max}) to the natural frequency $\omega_0 = 3$.

False.

(d) For a linear system $\vec{x}' = A\vec{x}$, $\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t}$ and $\vec{x}_2(t) = t\vec{v}_1 e^{\lambda_1 t}$ form a fundamental set of solutions if λ_1 is the repeated eigenvalue of A with one corresponding eigenvector $\vec{v}_1$.

False.

Question 3: **[14 marks]**

Consider the following initial value problem:

$$a_1 y''(t) + a_2 y'(t) + a_3 y(t) = f(t), \qquad y(0) = b_1,\ y'(0) = b_2, \qquad f(t) = \begin{cases} 0, & t < 1 \\ 17, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$

where a_1, a_2, a_3, b_1, and b_2 are constants.

(a) Use the method of Laplace Transforms to solve for $Y(s) = \mathcal{L}\{y(t)\}$ in terms of a_1, a_2, a_3, b_1, and b_2.

(b) Suppose $a_1 = 1$, $a_2 = -3$, $a_3 = 2$, $b_1 = b_2 = 0$. Calculate $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ (the solution to the initial value problem).

Solution:

(a) Express $f(t)$ in terms of step functions: $f(t) = 17[u(t-1) - u(t-2)] = 17[u_1(t) - u_2(t)]$. Apply Laplace transform on both sides of the eqn:

$$(a_1 s^2 + a_2 s + a_3)Y(s) = a_1 b_1 s + a_1 b_2 + a_2 b_1 + 17\left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right] \quad \Rightarrow$$
$$Y(s) = \frac{a_1 b_1 s + a_1 b_2 + a_2 b_1}{a_1 s^2 + a_2 s + a_3} + \frac{17(e^{-s} - e^{-2s})}{s(a_1 s^2 + a_2 s + a_3)}.$$

(b) For $a_1 = 1$, $a_2 = -3$, $a_3 = 2$, $b_1 = b_2 = 0$, we obtain

$$Y(s) = \frac{17(e^{-s} - e^{-2s})}{s(s^2 - 3s + 2)} = \frac{17(e^{-s} - e^{-2s})}{s(s-1)(s-2)} = 17(e^{-s} - e^{-2s})\left[\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}\right].$$

Using the partial fraction theorem, we obtain

$$A = \left[\frac{1}{(s-1)(s-2)}\right]_{s=0} = \frac{1}{2},\ B = \left[\frac{1}{s(s-2)}\right]_{s=1} = -1,\ C = \left[\frac{1}{s(s-1)}\right]_{s=2} = \frac{1}{2}.$$

Thus,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{17}{2}(e^{-s} - e^{-2s})\left[\frac{1}{s} - \frac{2}{s-1} + \frac{1}{s-2}\right]\right\}$$
$$= \frac{17}{2}\left[u_1(t)(1 - 2e^{t-1} + e^{2(t-1)}) + u_2(t)(1 - 2e^{t-2} + e^{2(t-2)})\right].$$

Question 4: **[14 marks]**

Consider the following system of nonhomogeneous equations

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

(a) Calculate the general solution given that $f_1(t) = 0$, $f_2(t) = 0$.

(b) Sketch the solution in part (a), analyze the stability and classify the critical point.

(c) Calculate the solution given that $f_1(t) = e^{-t}$, $f_2(t) = 1$.

Solution:

(a) $Tr = 2$, $Det = 1 - 4 = -3 \Rightarrow \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0 \Rightarrow \lambda_1 = -1$, $\lambda_2 = 3$.

For $\lambda_1 = -1$, $E_{\lambda_1} = \mathcal{N}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}_1$.

For $\lambda_2 = 3$, $E_{\lambda_1} = \mathcal{N}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_2$.

Therefore, $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_1\vec{v}_2 e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Y(t)\vec{c}.$

(b) See the figure. Thus $\vec{x} = \vec{0}$ is a saddle point and is unstable.

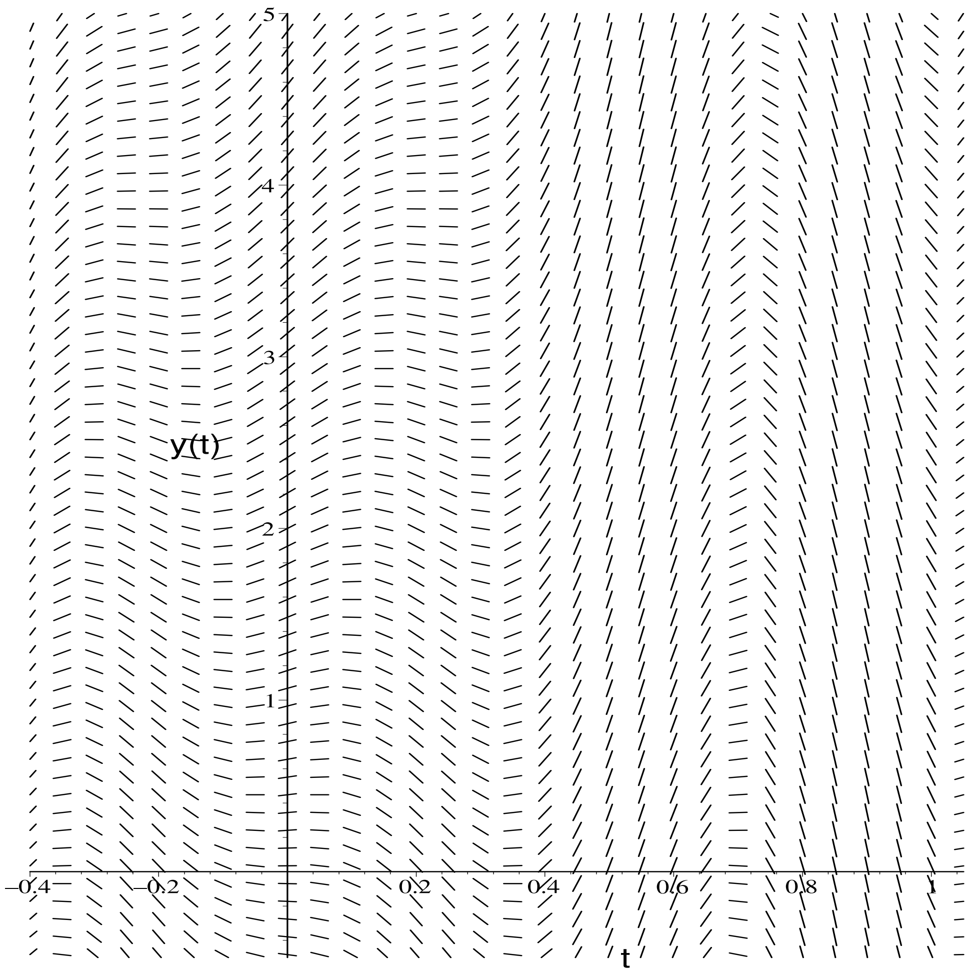
(c) For the nonhomogeneous system, $\vec{x}(t) = Y(t)[\vec{c} + \vec{u}(t)]$ where

$$\vec{u}(t) = \int Y^{-1}(t)\vec{b}dt = \int \frac{1}{2}\begin{bmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} dt = \frac{1}{2}\int \begin{bmatrix} 1 - e^{t} \\ e^{-4t} + e^{-3t} \end{bmatrix} dt = \frac{1}{2}\begin{bmatrix} t - e^{t} \\ -\frac{e^{-4t}}{4} - \frac{e^{-3t}}{3} \end{bmatrix}.$$

$$\vec{x}(t) = Y(t)[\vec{c} + \vec{u}(t)] = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{-t} - \tfrac{1}{8}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \tfrac{1}{3}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Question 5: **[12 marks]**

Consider the following slope field of the differential equation $\frac{dy}{dt} = f(t, y)$.

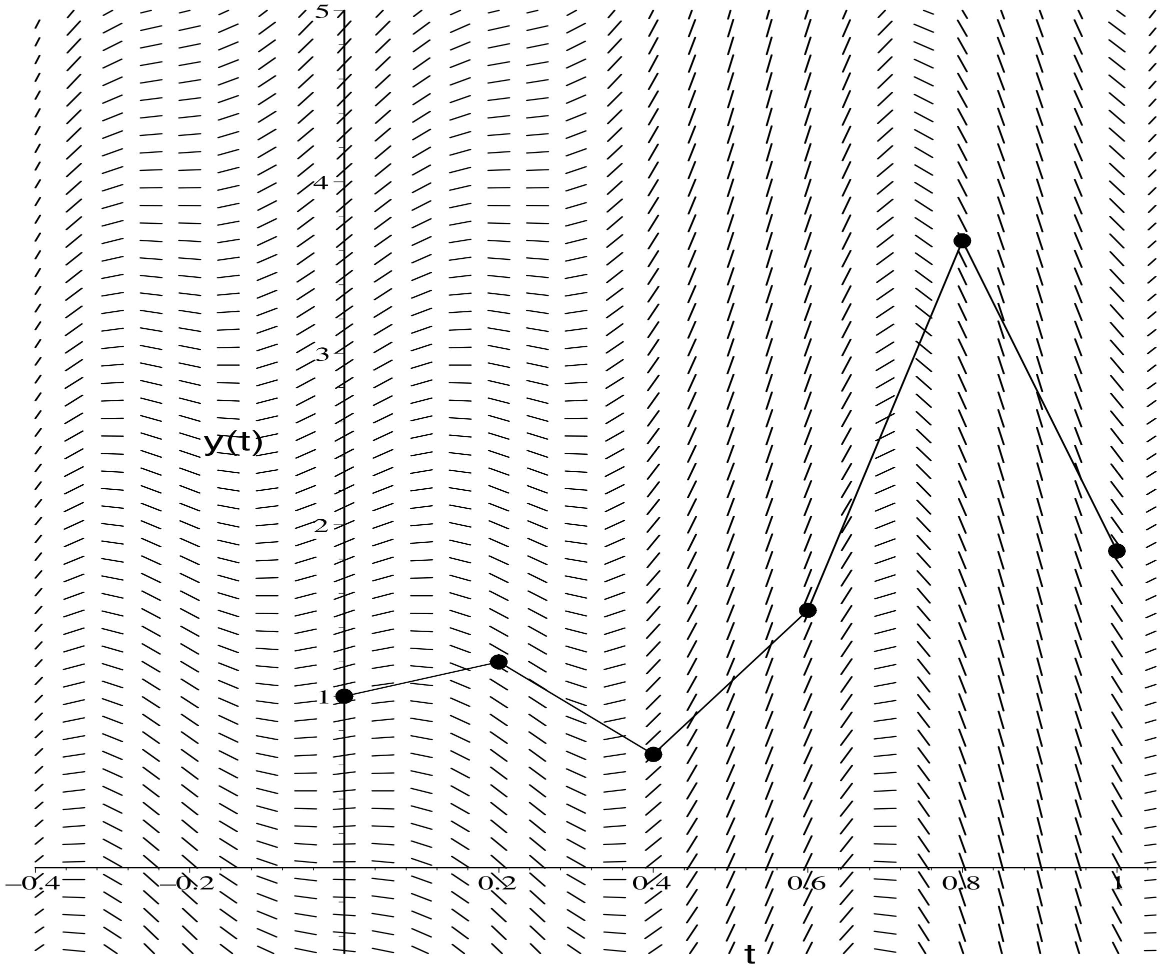


(a) Euler's method with step size $h = 0.2$ is being used to estimate $y(1)$. Let $y(0) = 1$, $t_n = nh$ for $n = 1, 2, 3, \ldots, 5$ and y_n be the estimate of $y(t_n)$. Plot the 6 points $p_n = (t_n, y_n),\ n = 0, \ldots, 5$ on the graph.

(b) The table of values show the estimates of $y(2)$ for various step sizes using Euler's method and Improved Euler's method. Assume the magnitude of the error for each method has the form Kh^p, where h is the step size used and K, p are constants. Estimate the largest step size h so that the error using the improved Euler method is accurate to within $\pm 10^{-2}$.

step size h	Euler's method estimate	Improved Euler estimate
0.3	271.75	291.50
0.1	291.95	299.50

Solution:
(a)



(b)

$$\begin{aligned} (1) \quad & y(2) = 291.50 + K(0.3)^2 + O(h^3) \\ (2) \quad & y(2) = 299.50 + K(0.1)^2 + O(h^3) \end{aligned}$$

$$(1) - (2) \quad \Rightarrow \quad 0 = -8 + K(0.08) \quad \Rightarrow \quad K = 100$$

The error using improved Euler's method is $E = Kh^2 + O(h^3)$, where $K = 100$. We want $E \leq 10^{-2}$ so:

$$100h^2 \leq 10^{-2} \quad \Rightarrow \quad h^2 \leq \frac{10^{-2}}{100} \quad \Rightarrow \quad h^2 \leq 10^{-4} \quad \Rightarrow \quad h \leq 10^{-2}.$$

So the largest step size that can be used is $\boxed{h = 0.01}$.

Question 6: **[16 marks]**

These questions do not require lengthy calculations. **Put your answers in the boxes.**

(a) For what value of the real parameter a is the following equation exact? Find the general solution for that value of a. $e^{ay} + (a^2 - xe^{-y} + \sin y)\dfrac{dy}{dx} = 0$?

$a = -1; \quad y - \cos y + xe^{-y} = C.$

(b) For the nonhomogeneous equation $y'' + 4y' + 13y = e^{-2t}\sin(3t)$, write down the correct form of a particular solution that contains the undetermined coefficients (do not calculate those coefficients).

$y_p(t) = t[A\cos(3t) + B\sin(3t)]e^{-2t}.$

(c) Given that $y_1(t) = t$ is a solution to $t^2y'' + ty' - y = 0,\ t > 0$, find a second solution that is linearly independent of $y_1(t)$.

$y_2(t) = t^{-1}$ or $\frac{c}{t}$ for any constant $c \neq 0$.

(d) Express the solution of the initial value problem $y'' + 2y' + y = h(t),\ y(0) = y'(0) = 1$ in a form that contains $h(t)$ in an integral.

$y(t) = e^{-t} + 2te^{-t} + \int_0^t \tau e^{-\tau} h(t-\tau)d\tau$ or $e^{-t} + 2te^{-t} + \int_0^t (t-\tau)e^{-(t-\tau)}h(\tau)d\tau$

Question 7: **[20 marks]**

For the following two different problems, you need to choose one and only one to solve. Credit will be given to the solution of either 7.I or 7.II but not to both!

7.I Consider the following predator-prey model, given by

$$\begin{aligned} x' &= x(1 - \gamma x - 0.5y) \\ y' &= y(-0.25 + 0.5x). \end{aligned}$$

Let $\gamma = 0.5$. in parts (a), (b), (c) and (d).

(a) Find all the critical points (steady states) of this system of differential equations. Represent these points with a black dot in the phase plane provided below.

(b) For each critical point, discuss its local stability properties and classify its type. Determine the trajectory flows near each one of these points.

(c) Draw a rough phase portrait for the system using the provided phase plane (please try it on a separate draft before putting down the final picture).

(d) Determine the long-term or limiting behaviour of $x(t)$ and $y(t)$ as $t \to \infty$ for any initial condition $x(0), y(0) > 0$. Interpret the results in terms of the populations of the two species.

(e) For $\gamma = 0$, determine how the number, location, and type of the critical points change. Without attempting a detailed analysis, what can you say about the change in the limiting behaviour of $x(t)$ and $y(t)$ for $x(0), y(0) > 0$?

Solution:

(a) In order to find the critical points, we shall find the x- and y-nullclines.
The x-nullclines are: $x = 0$ (line) and $y = 2 - x$ (line).
The y-nullclines are: $y = 0$ (line) and $x = 1/2$ (line).
Thus the critical points in this case are: $(0, 0)$, $(2, 0)$ and $(1/2, 3/2)$.
See the sketch for plotting these points.

(b) Let's begin first by finding the Jacobian matrix:

$$J = \begin{pmatrix} 1 - x - 0.5y & -0.5x \\ 0.5y & -0.25 + 0.5x \end{pmatrix}$$

(i) At $(0, 0)$:

$$J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -0.25 \end{pmatrix}.$$ The eigenvalues are $m_1 = 1 > 0$ and $m_2 = -0.25 < 0$.

Therefore $(0, 0)$ is an unstable saddle point. The eigenvectors are $(1, 0)^T$ and $(0, 1)^T$.

(ii) At $(2, 0)$:

$$J_{(2,0)} = \begin{pmatrix} -1 & -1 \\ 0 & 0.75 \end{pmatrix}.$$ The eigenvalues are $m_1 = -1 < 0$ and $m_2 = 0.75 > 0$.

Therefore $(2, 0)$ is an unstable saddle point. The eigenvectors are $(1, 0)^T$ and $(1, -1.75)^T$.

(ii) At $(1/2, 3/2)$:

$$J_{(1/2,3/2)} = \begin{pmatrix} -1/4 & -1/4 \\ 3/4 & 0 \end{pmatrix}.$$ The eigenvalues are $m_1 = \frac{-1}{8} + i\frac{\sqrt{11}}{8}$ and $m_2 = \frac{-1}{8} - i\frac{\sqrt{11}}{8}$.

We have two complex conjugate roots with negative real parts. Therefore $(1/2, 3/2)$ is an asymptotically stable spiral sink.

(c) See the sketch of phase portrait in the figure.

(d) Since the critical point $(1/2, 3/2)$ is the only asymptotically stable critical point, the long term behavior of $\mathbf{x} = (x, y)^T$ will approach that equilibrium solution. In other words, $\lim_{t\to\infty} \mathbf{x}(t) = (1/2, 3/2)$. This means that the two species involved in this predator-prey model will both survive and their populations will gradually approach $(1/2, 3/2)$. The solutions curves for x and y will oscillate with decaying amplitudes toward that equilibrium solution.

(e) When $\gamma = 0$, then one critical point will disappear and end up with only two of them. These two critical points are $(0, 0)$ and $(1/2, 2)$ (the y-nullcline $y = 2 - x$ in the previous case becomes $y = 2$). The current model is identical to the standard simple predator-prey model we are familiar with. The critical point $(0, 0)$ stays as an unstable equilibrium solution, but the critical point $(1/2, 2)$ is a stable center. The phase portrait will show a closed curve reflecting an oscillatory behavior in both populations, x and y, with constant amplitudes for all time.

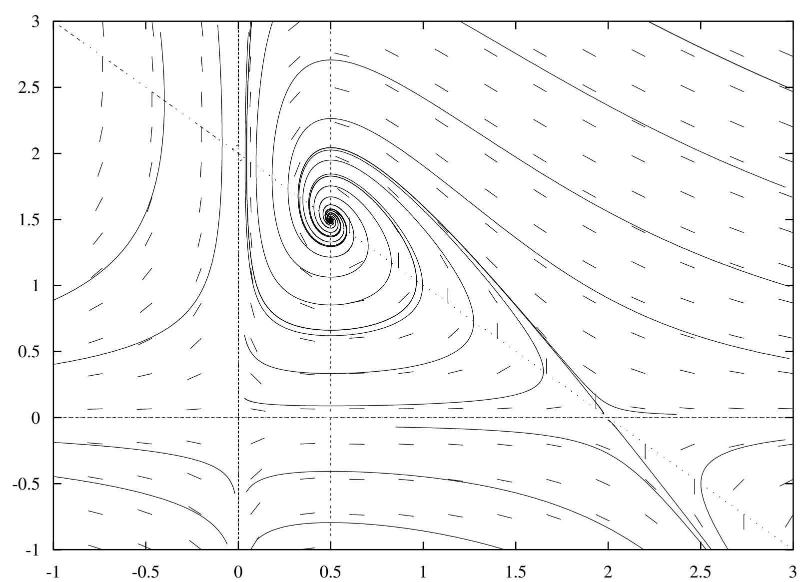


Figure 1: Phase portrait of the system.

7.II A mass-spring system is governed by the following initial value problem

$$my''(t) + \gamma y'(t) + ky(t) = A\sin(\omega t), \qquad y(0) = 0,\ y'(0) = -4$$

where the positive constants m, γ, k, A and ω are the mass, damping coefficient, spring constant, forcing amplitude and forcing frequency, respectively.

(a) Suppose the system is undamped. Determine the constraints (if any) on the constants m, γ, k, A and ω so that the solution exhibits pure resonance.

(b) Determine an approximate value of the resonant frequency given that $m = 2$, $k = 5$, $A = 7$, and $\gamma << 1$.

(c) Sketch the steady-state solution to the initial value problem given $m = 1$, $\gamma = 0$, $k = 16$, $A = 36$, and $\omega = 5$.

(d) Suppose the general solution can be written in the form

$$y(t) = A_0 e^{-\gamma t/(2m)} \cos(\omega_0 t + \delta) + A_1 \cos(\omega t + \phi)$$

Determine the constraints (if any) on the constants m, γ, A_0, A_1, ω_0, ω, δ, ϕ so that the solution exhibits the phenomenon of beat (*beat* refers to a high frequency vibration with an amplitude that varies periodically with a much slower frequency).

Solution:

(a) Since the driving term is periodic, pure resonance will be observed when the driving frequency ω is equal to the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$, i.e. $\boxed{\omega = \sqrt{\frac{k}{m}}}$. This implies that $\boxed{\gamma = 0}$ and the homogeneous solution y_h can be written in the form: $y_h(t) = A_0 \cos(\omega_0 t + \delta)$. The driving amplitude $A \neq 0$.
(b) The resonant frequency yields the maximum steady state amplitude and occurs at about $\omega \approx \omega_0 = \sqrt{5/2}$.
(c) The characteristic equation is $\lambda^2 + 16 = 0$, so $\lambda = \pm 4i$ and the homogeneous solution has the form:

$$y_h(t) = C_1 \cos(4t) + C_2 \sin(4t)$$

The particular solution has the form $y_p(t) = D_1 \cos(5t) + D_2 \sin(5t)$ and substituting y_p into the DE gives:

$$y'' + 16y = 36\sin(5t) \ \Rightarrow \ \Big[-25D_1\cos(5t) - 25D_2\sin(5t)\Big] + \Big[16D_1\cos(5t) + 16D_2\sin(5t)\Big] = 36\sin(5t)$$

$$\Rightarrow \quad -9D_1\cos(5t) - 9D_2\sin(5t) = 36\sin(5t) \quad \Rightarrow \quad D_1 = 0 \text{ and } D_2 = -4.$$

So the general solution has the form:

$$y(t) = y_h(t) + y_p(t) = C_1\cos(4t) + C_2\sin(4t) - 4\sin(5t)$$

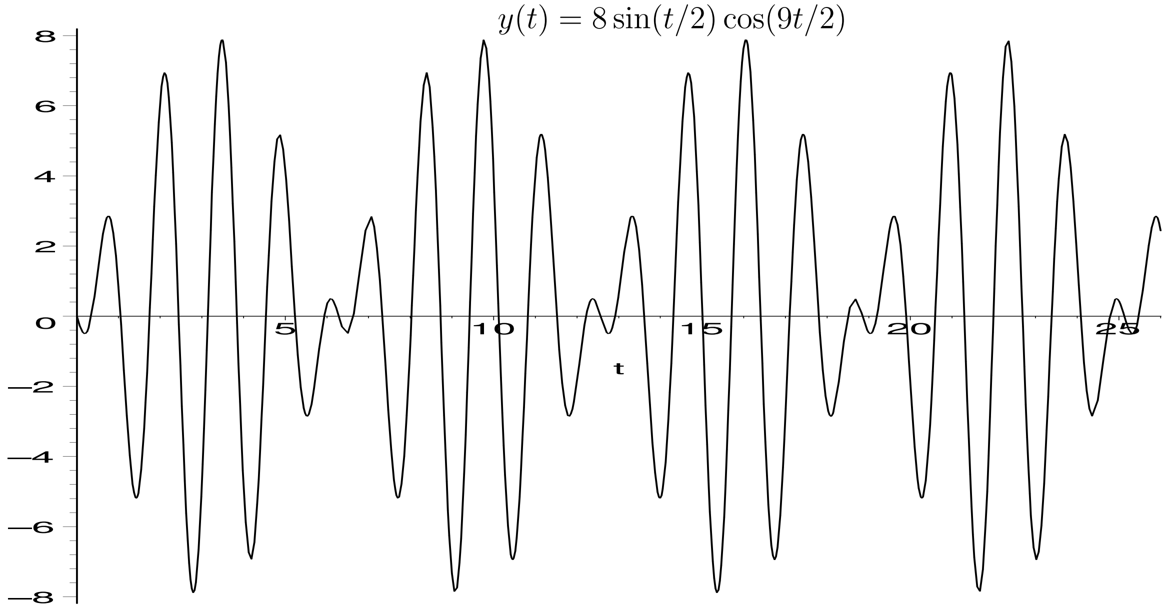
$$\Rightarrow \quad y'(t) = -4C_1\sin(4t) + 4C_2\cos(4t) - 20\cos(5t)$$

Applying the initial condition $y(0) = 0 \Rightarrow C_1 = 0$ and $y'(0) = -4 \Rightarrow C_2 = 4$. Therefore

$$y(t) = 4\sin(4t) - 4\sin(5t)$$

It is easier to sketch the solution if we apply the trig identity $\sin(a-b) - \sin(a+b) = 2\sin(b)\cos(a)$. This implies $\sin(c) - \sin(d) = 2\sin\left(\frac{d-c}{2}\right)\cos\left(\frac{c+d}{2}\right)$ so

$$y(t) = 8\sin(t/2)\cos(9t/2)$$



(d) The forcing term is periodic, so in order for the solution to exhibit the phenomenon of beats, the homogeneous solution must be periodic, i.e. $\boxed{\gamma = 0}$. In this case

$$y_h = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$. The particular solution can be put in the form $y_p = t^s\left(A_1\cos(\omega t + \phi)\right)$, but we want $s = 0$, so $\boxed{\omega \neq \omega_0 = \sqrt{k/m}}$. The general solution is then

$$y(t) = y_h(t) + y_p(t) = A_0\cos(\omega_0 t + \delta) + A_1\cos(\omega t + \phi).$$

The phenomenon of beats will be observed whenever $\boxed{|A_0| = |A_1|}$.

The phase angles $\boxed{\delta \text{ and } \phi \text{ are unconstrained}}$.