MATH 221 - FINAL EXAM

University of British Columbia

December 15, 2015.

Last name: ____________

First name: ____________

Section number:

- 102 - MWF 10-11 am - Justin Tzou
- 103 - MWF 1-2 pm - Daniel Coombs
- 104 - MWF 1-2 pm - Miljan Brakocevic

↑ Fill in bubble corresponding to your section.

← Fill in bubble-number below each number in your student number.

Student number:

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9

This page will be removed for marking and data entry so you must put your personal information on the next page as well.
DO NOT REMOVE THIS PAGE YOURSELF.

For marking purposes - DO NOT WRITE IN THIS BOX:

Question:	Total mark:												
1	0	1	2	3	4	5							
2	0	1	2	3	4	5	6	7	8	9	10		
3	0	1	2	3	4	5							
4	0	1	2	3	4	5							
5	0	1	2	3	4	5							
6	0	1	2	3	4	5	6	7	8	9	10	11	12
7	0	1	2	3	4	5	6	7	8	9	10	11	12
8	0	1	2	3	4	5	6	7	8	9	10	11	12
9	0	1	2	3	4	5	6	7	8	9	10	11	12
10	0	1	2	3	4	5	6	7	8	9	10		
11	0	1	2	3	4	5	6	7	8	9	10	11	12

The University of British Columbia

Final Examination - December 15th, 2015

Mathematics 221

Time: 2 hours

Last Name ____________________ **First** ____________ **Signature** ____________________

Student Number ____________________

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- Show your work. A correct answer without intermediate steps will receive no credit.
- This exam has 12 pages, including both cover sheets.

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;

(b) purposely exposing written papers to the view of other candidates or imaging devices;

(c) purposely viewing the written papers of other candidates;

(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		5
2		10
3		5
4		5
5		5
6		12
7		12
8		12
9		12
10		10
11		12
Total		100

Prob. 1. (5pts) CIRCLE THE CORRECT ANSWER. No work need be shown for this question.

(a) True/False: For any pair of square matrices A and B, $(A+B)(A-B) = A^2 - B^2$.

TRUE	FALSE

(b) True/False: If the determinant of a matrix is zero, then the matrix has at least one column that is a scalar multiple of another column.

TRUE	FALSE

Prob. 2. (10pts) Consider the following system of linear equations.

$$\begin{aligned} x - y + 2z &= 1 \\ x + y + az &= 1 \\ -3x + 6y - 7z &= b \end{aligned}$$

(a) For what values of a and b is the system consistent? Show your work.

(b) For what values of a and b does the system have exactly one solution? Show your work.

(c) For what values of a and b does the system have infinitely many solutions?

Prob. 3. (5pts) Let T be a linear transformation of $\mathbb{R}^2$ defined as follows: First, rotate π radians counterclockwise about the origin. Second, reflect in the line $y = x$. Find the standard matrix for this transformation. Explain your reasoning.

Prob. 4. (5pts) Find the determinant of the matrix $A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 3 & 217 & 0 & -342 & 0 \\ 0 & 9 & 0 & 19 & 1 \\ 0 & -12 & 0 & 14 & 0 \\ 0 & 12 & -3 & 1999 & 0 \end{pmatrix}$. Show your reasoning.

Prob. 5. (5pts) Vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^{21}$ are found such that $\vec{x}\cdot\vec{x} = 10$, $\vec{x}\cdot\vec{y} = 1$ and $\vec{y}\cdot\vec{y} = 3$.

(a) Find $||2\vec{x} - 3\vec{y}||$. Show your work.

(b) Define $W = \text{span}\{\vec{x}, \vec{y}\}$. What is the dimension of $W^{\perp}$? Briefly explain your answer.

Prob. 6. (12pts) Consider the non-zero $n \times n$ matrix $A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ n & n & \dots & n \end{pmatrix}$.

(a) Find all eigenvalues of A and specify their multiplicities. Explain your reasoning.

(b) Find the eigenvector of A corresponding to its largest eigenvalue. Show all work.

Prob. 7. (12pts)

(a) Find two vectors that span the plane P in $\mathbb{R}^3$ defined by $x_1 + x_2 + x_3 = 0$. Show your work.

(b) Find the orthogonal projection of the vector $\vec{b} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ onto the plane P. Show your work.

(c) Find the distance from $\vec{b}$ to the closest point on the plane P. Show your work.

(d) Let $\mathbf{M}$ be the 3×3 projection matrix that projects vectors in $\mathbb{R}^3$ onto the plane P. Determine the eigenvalues of $\mathbf{M}$ and specify their multiplicities. This can be done without explicitly constructing $\mathbf{M}$. Explain your answer.

(e) Find the rank of $\mathbf{M}$ and the dimension of the null space of $\mathbf{M}$. This can be done without explicitly constructing $\mathbf{M}$. Write the answer in the box and also describe your reasoning.

rank($\mathbf{M}$)= dim(Nul($\mathbf{M}$))=

Prob. 8. (12pts) A population of fish has three classes: juveniles, young adults and old adults.

Let the population in year n be given by the vector $\vec{x}_n = \begin{pmatrix} j_n \\ y_n \\ o_n \end{pmatrix}$ where j_n, y_n and o_n indicate juveniles, young adults and old adults at year n, respectively.

Each year, one third of the juveniles mature to become young adults in the following year, while three-quarters of the young adults survive and become old adults the following year. Old adults never survive to the next year. Each young adult has M offspring each year and each old adult has 1 offspring each year.

(a) Write the population dynamics in matrix vector form $\vec{x}_{n+1} = A\vec{x}_n$ where A is a 3×3 Leslie matrix that you must write explicitly.

(b) Find a number of offspring M so that the matrix A has one of its eigenvalues equal to 1. Show your work.

$M =$

(c) For this value of M, find the eigenvector corresponding to the eigenvalue $\lambda = 1$. Show your work.

(d) Explain the meaning of this eigenvalue and eigenvector in biological terms.

Prob. 9. (12 pts) (a) Define the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Find the diagonalization of A, that is, find D and P such that $A = PDP^{-1}$ where D is a diagonal matrix. Show your work.

$D =$	$P =$

(b) Calculate A^{2015}. Show your work.

$A^{2015} =$

Prob. 10. (10 pts) The following commands were typed into MATLAB, followed by `<enter>`.

```
clear all;
A = [1,2;3,4];
x = [1;1];
y = [1;-1];
```

What will be the result of the following MATLAB commands, followed by `<enter>`?
If the command produces an error message, state the reason.

(a) `A*x`

(b) `y'*A*x`

(c) `x'*y`

(d) `A*y'`

(e) `x.*y`

Prob. 11. (12 pts) Let $\vec{u}, \vec{v}, \vec{w}$ be three distinct vectors in $\mathbb{R}^3$. Prove that if $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis of $\mathbb{R}^3$ then $\{\vec{u} + 2\vec{v} + 3\vec{w}, 2\vec{v} + 3\vec{w}, 3\vec{w}\}$ is also a basis of $\mathbb{R}^3$.

END OF EXAM