Final Exam

December 21, 2009

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (6 points)
Let V and W be subspaces of $\mathbb{R}^4$. Assume that $\dim V = 2$ and $\dim W = 3$.

(a) What are the possible dimensions of $V \cap W$?
(b) For each of these dimensions, give explicit examples of V and W, where this dimension is achieved (either in terms of equations defining V and W, or in terms of generating sets for V and W).
(c) Explain why no other dimensions are possible.

Problem 2. (6 points)
Find out if the following vectors are linearly dependent. If they are, express one of them as a linear combination of the others.

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix} \qquad \vec{v}_4 = \begin{pmatrix} -2 \\ -5 \\ 3 \\ -7 \end{pmatrix}$$

Problem 3. (6 points)
Find the determinant of the $n \times n$-matrix A_n, whose entries along the diagonal are all equal to 3, whose entries on the subdiagonal are all equal to -2, and whose entries on the superdiagonal are all equal to 2. All other entries of A_n are zero.

$$A_n = \begin{pmatrix} 3 & 2 & & & \\ -2 & 3 & 2 & & \\ & -2 & \ddots & & \\ & & & 3 & 2 \\ & & & -2 & 3 \end{pmatrix}$$

Write $a_n = \det(A_n)$.

(a) Use Laplace expansion to write down a recursion for a_n, which expresses a_n in tems of a_{n-1} and a_{n-2}.
(b) Introduce an auxiliary variable $b_n = a_{n-1}$, to turn the recursion into a 2×2 discrete dynamical system.
(c) Solve this system by using the eigenvalue method.

Problem 4. (6 points)
The reflection $R_{\vec{u}} : \mathbb{R}^n \to \mathbb{R}^n$ across the hyperplane orthogonal to the unit vector $\vec{u} \in \mathbb{R}^n$ is given by the formula

$$R_{\vec{u}}(\vec{v}) = \vec{v} - 2\langle \vec{v}, \vec{u} \rangle \, \vec{u} \, ,$$

for all $\vec{v} \in \mathbb{R}^n$.

(a) Find the matrix of $R_{\vec{u}}$ in standard coordinates, if $\vec{u} \in \mathbb{R}^3$ is in the direcction of $\left(\begin{smallmatrix} 1 \\ 2 \\ -1 \end{smallmatrix}\right)$.

(b) Explain why, no matter in what direction $\vec{u}$ points, you will always get a symmetric matrix for $R_{\vec{u}}$.

Problem 5. (6 points)
Consider the following matrix:

$$A = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

The matrix A is orthogonal, and has determinant 1. Therefore, A describes a rotation about an axis through the origin.

(a) Find the rotation axis.

(b) Find the rotation angle θ, where $0 \leq \theta \leq \pi$.

Problem 6. (6 points)
Find out if any of the following three matrices are similar to each other:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Justify your answer (for example, by appealing to the theorem on Jordan canonical forms).

Problem 7. (6 points)

(a) Find the principal axes of the quadric surface in $\mathbb{R}^3$ given by

$$y^2 + xz = 1 \, .$$

(b) Sketch this quadric surface.

Problem 8. (6 points)
Suppose a real 3×3-matrix A has characteristic polynomial

$$(t-2)(t-3)(t-5) = t^3 - 10t^2 + 31t - 30\,.$$

(a) Prove that if $\vec{v}$ is an eigenvector of A, then

$$(A^3 - 10A^2 + 31A - 30I_3)\,\vec{v} = \vec{0}\,.$$

(b) Prove that if $\vec{v}$ is an arbitrary vector in $\mathbb{R}^3$, then

$$(A^3 - 10A^2 + 31A - 30I_3)\,\vec{v} = \vec{0}\,.$$

(c) Deduce that A satisfies the matrix equation

$$A^3 - 10A^2 + 31A - 30I_3 = 0\,.$$