Math 256 – Final examination

University of British Columbia

April 21, 2016, 3:30 pm to 6:00 pm

Last name (print): First name:

ID number:

This exam is "closed book" with the exception of a single 8.5"x11" formula sheet (hand-written, one-sided, no photocopies). Calculators or other electronic aids are not allowed.

Page	Score	Max
2		8
3		4
4		7
5		13
6		12
7		9
8		10
Total		63

Multiple choice

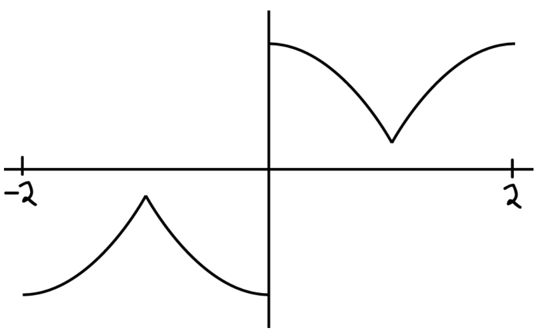
1. [**4 pts**] For each of the following equations, choose the best form for $y_p(t)$ for the Method of Undetermined Coefficients from the list below by placing the appropriate letter (a, b, c...) in the box beside it.

   - ☐ $y'' + y' - 12y = 2e^{3t}$
   - ☐ $y'' - 3y' + 2y = 9t^3e^{3t}$
   - ☐ $y'' - 6y' + 9y = -4te^{3t}$
   - ☐ $y'' - 4y = 3t^2e^{3t}$

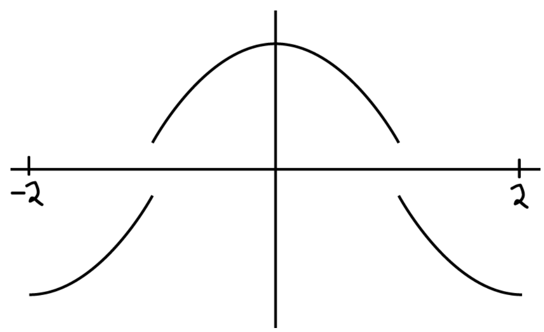
   (a) $y_p(t) = Ae^{3t}$
   (b) $y_p(t) = Ate^{3t}$
   (c) $y_p(t) = (At + B)e^{3t}$
   (d) $y_p(t) = At^2e^{3t}$
   (e) $y_p(t) = (At^2 + Bt + C)e^{3t}$
   (f) $y_p(t) = At^3e^{3t}$
   (g) $y_p(t) = (At^3 + Bt^2)e^{3t}$
   (h) $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{3t}$

2. [**2 pts**] Which of the following functions has a Fourier series with only sine terms of the form $b_n \sin(n\pi x/2)$ with $b_n = 0$ for all n even?

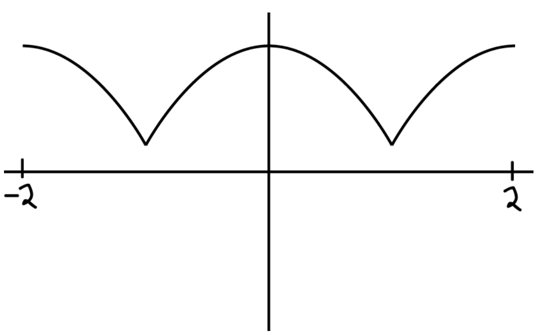
(a)



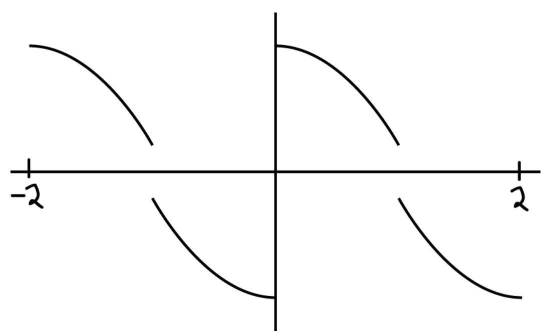
(b)



(c)



(d)



3. [**2 pts**] The solution to a second order linear homogeneous differential equation with constant coefficients has one non-zero solution for which $\lim_{t\to\infty} x_1(t) = -\infty$, another non-zero solution for which $\lim_{t\to\infty} x_2(t) = 0$ and a third non-zero solution for which $\lim_{t\to\infty} x_3(t) = \infty$. Which of the following could be the differential equation in question?

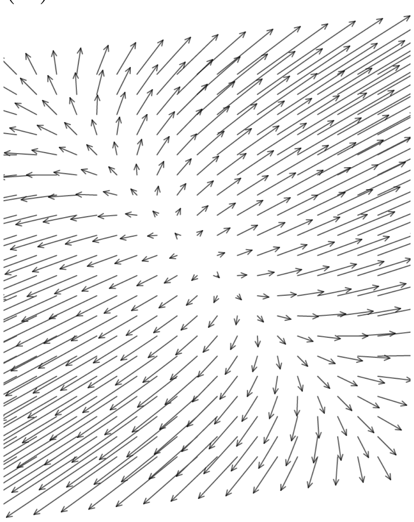
   (a) $ay'' + by' + cy = 0$ where $a > 0$, $b > 0$, and $c > 0$.
   (b) $ay'' + by' + cy = 0$ where $a > 0$, $b > 0$, and $c < 0$.
   (c) $ay'' + by' + cy = 0$ where $a > 0$, $b < 0$, and $c > 0$.
   (d) $ay'' + by' + cy = 0$ where $a < 0$, $b > 0$, and $c < 0$.

4. **[2 pts]** Which of the following pairs of functions are not independent?

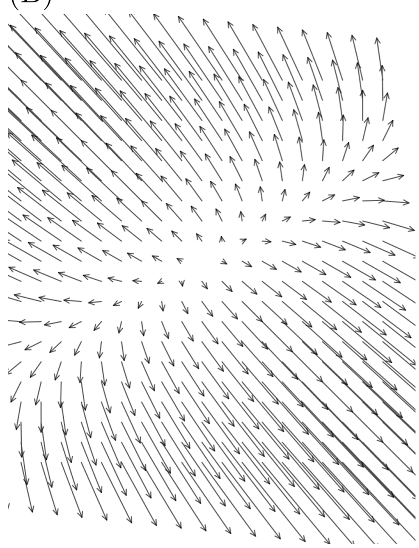
(a) $f(t) = 3t, \quad g(t) = 3t - 4$
(b) $f(t) = (3t)^2, \quad g(t) = (3t-4)^2$
(c) $f(t) = e^{3t}, \quad g(t) = e^{3t-4}$
(d) $f(t) = \cos(3t), \quad g(t) = \cos(3t-4)$

5. **[2 pts]** Choose the correct matches between the vector fields (A-D) and solutions (i-ii).

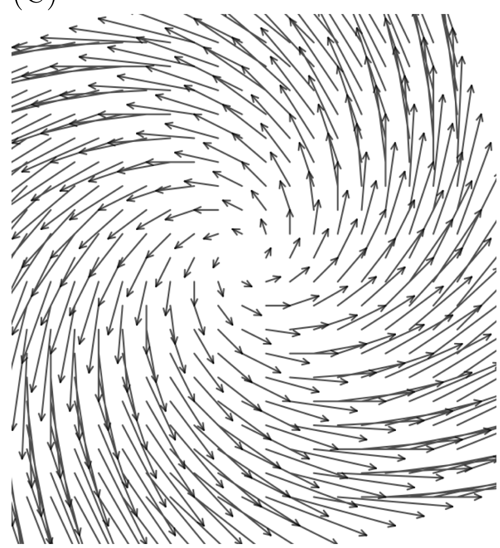
(A)



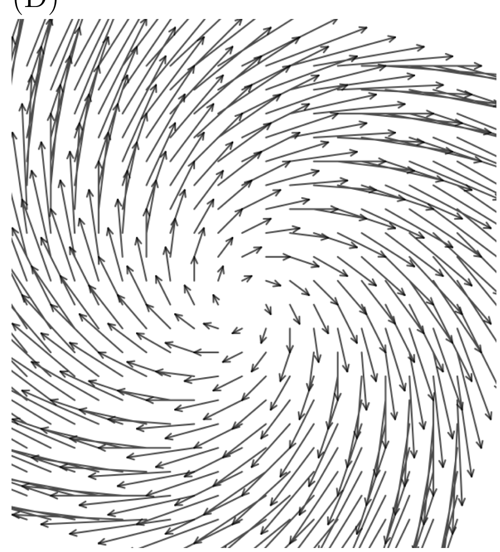
(B)



(C)



(D)



(i) $\mathbf{x(t)} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

(ii) $\mathbf{x(t)} = e^{2t} \left[C_1 \begin{pmatrix} \cos(4t) \\ -\sin(4t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(4t) \\ \cos(4t) \end{pmatrix} \right]$

Circle one:

(a) (i) – (A); (ii) – (C)
(b) (i) – (A); (ii) – (D)
(c) (i) – (B); (ii) – (C)
(d) (i) – (B); (ii) – (D)

Written answers

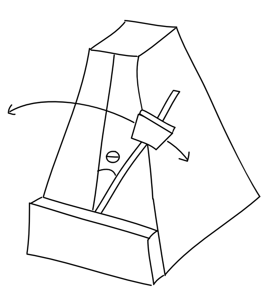
6. **[7 pts]** The angular deflection, $\theta(t)$, of a metronome (an "upside-down pendulum" used to set a beat for musicians - see sketch below) satisfies the equation $\theta''+15\theta = 0$ in the absence of forcing. A truck driving by outside induces an external force on the metronome given by $F(t) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nt)$.

   (a) What is the period of the metronome when released from a non-zero angle in the absence of forcing?

   Unforced period = $\boxed{}$

   (b) Find a Fourier series representation of the particular solution $\theta(t)$ when the metronome is subject to forcing.

   (c) What is the period of the largest amplitude sin/cosine function in the Fourier series of the solution?

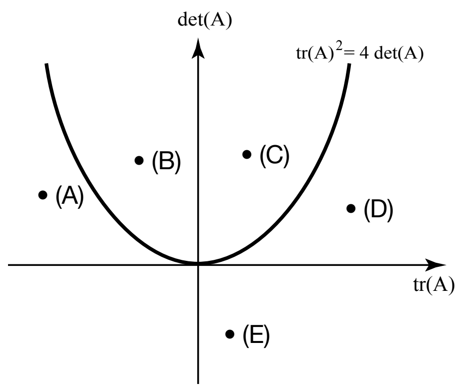


   Period = $\boxed{}$

7. [**7 pts**] Consider the system of equations given in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha+1 & 1 \\ 1 & \alpha-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Using the letters from the diagram below, list (in order) the regions of the trace/determinant plane that the system moves through as α goes from $-\infty$ to ∞.



Regions:

(b) Find the value of alpha at each transition.

8. [**6 pts**] Solve the Diffusion equation $u_t = 2u_{xx}$ with boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(3,t) = 2$ and initial condition $u(x,0) = 2x + 5\cos(7\pi x/3)$.

9. **[12 pts]** Consider the function $m(t) = 24 - 18(u_1(t) - u_2(t))$.

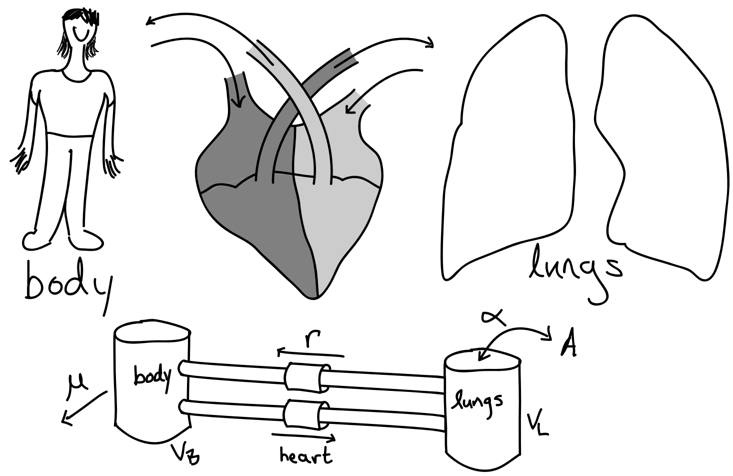
(a) Sketch $m(t)$.

(b) Solve the differential equation $y' + 6y = m(t)$ with initial condition $y(0) = 0$.

(c) Sketch the solution to the equation in (b).

10. **[5 pts]** Oxygenated blood (light grey) is pumped from the lungs through the heart to the body at a rate of $\boxed{r = 5\ell/\text{min}}$. Deoxygenated blood (dark grey) is pumped from the body through the heart to the lungs at the same rate.

Oxygen (O_2) is transferred between the air and the blood in the lungs at a rate proportional to the difference between air and blood concentrations in mol/ℓ with proportionality constant $\boxed{\alpha = 3\ell/\text{min}}$. The concentration of O_2 in air is a constant $\boxed{A \text{ (in mol}/\ell)}$. O_2 is removed from the body by metabolism at a rate proportional to the number of mol of O_2 in the body with a rate constant $\boxed{\mu = 2/\text{min}}$. The volume of blood in the body is $\boxed{V_B = 5\ell}$ and the volume of blood in the lungs is $\boxed{V_L = 2\ell}$. Write down a system of differential equations for the number of mol of O_2 in the body, $B(t)$, and in the lungs, $L(t)$.



11. **[4 pts]** Calculate the Fourier series for the function $f(x) = \delta(x-1) - \delta(x+1)$ when $-2 \le x \le 2$ and is periodic beyond that interval.

12. [**10 pts**] The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{4}{(s+1)(s^2+2s+5)} + \frac{3}{s+1}.$$

(a) Find the solution $y(t)$ by inverting $Y(s)$.

(b) Provide a differential equation and initial value(s) that have a transformed solution $Y(s)$.

This page is for rough work and doodles. It will not be marked.