Math 257 Final Exam

April 20, 2010

Duration: 2 hours 30 minutes

Last Name: ____________ First Name: __________ Student Number: ______

Do not open this test until instructed to do so. Relax. This exam should have 11 pages, including this cover sheet. It is a closed book exam; no textbooks, calculators, laptops, formula sheets or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Please explain your work, and circle your final solutions.** Use the extra pages if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1 (20 points)

Consider the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + \left(2x - x^2\right)\frac{dy}{dx} + \left(\frac{3}{2}x - \frac{3}{4}\right)y = 0.$$

(a) [5] Classify the point $x = 0$ as an ordinary point, regular singular point or irregular singular point.

(b) [10] By seeking solutions of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, find the roots of the indicial equation and find the recurrence relation for a_n depending on r.

(c) [5] Find the solution for the larger of the two values of r, taking $a_0 = 1$.

Extra page (Problem 1)

Problem 2 (20 points)

Consider the heat equation with a source

$$u_t = u_{xx} + \sin x, \qquad 0 < x < \pi,\ t > 0,$$

$$u(0,t) = 0, \quad u(\pi,t) = 1, \qquad u(x,0) = 0.$$

(a) [6] Find the steady state solution $u_\infty(x)$.

(b) [6] Show that the Fourier sine coefficients for $u_\infty(x)$ are

$$b_n = \begin{cases} 1 + \frac{2}{\pi} & n = 1 \\ -\frac{2}{n\pi}(-1)^n & n \geq 2 \end{cases}$$

(c) [8] By writing $u(x,t) = u_\infty(x) + v(x,t)$ and separating variables, determine the solution for $u(x,t)$.

Extra page (Problem 2)

Problem 3 (20 points)

(a) [10] Waves on a string that is fixed at ends $x = 0$ and $x = 8$ satisfy

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < 8,\ t > 0,$$

$$u(0,t) = 0, \quad u(8,t) = 0,$$

with initial conditions

$$u(x,0) = f(x) = \begin{cases} 0 & 0 < x < 3 \\ 1 & 3 \le x \le 5 \\ 0 & 5 < x < 8 \end{cases}, \qquad u_t(x,0) = 0.$$

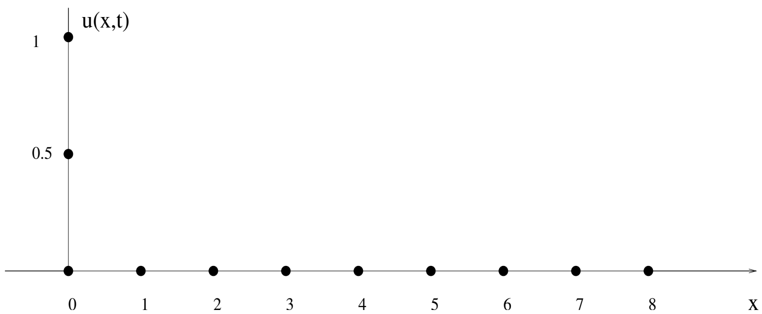
Use separation of variables to find the solution for $u(x,t)$ in terms of the Fourier sine coefficients b_n for $f(x)$.

You are not *required to actually calculate the Fourier coefficients, which are*

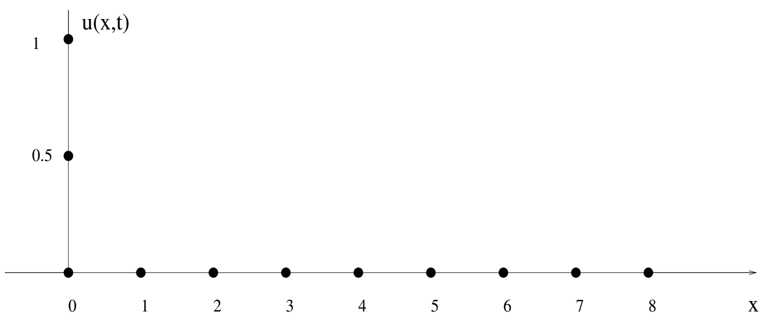
$$b_n = \begin{cases} \frac{4}{n\pi} \cos\left(\frac{3}{8} n\pi\right) & n \textit{ odd} \\ 0 & n \textit{ even} \end{cases}$$

(b) [10] For wave speed $c = 1$, on the axes on the next page sketch the solution at each of the times $t = 0$, 1, 2 and 4.

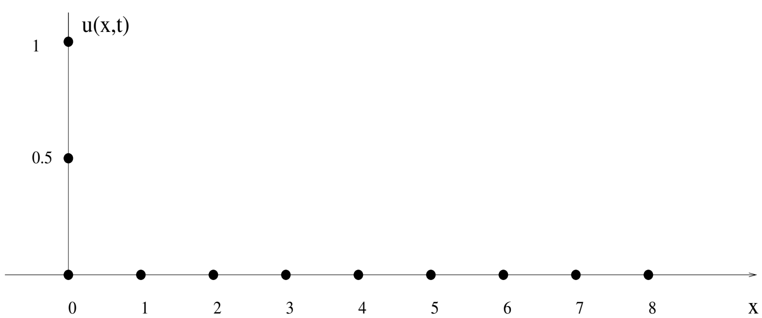
(Problem 3)



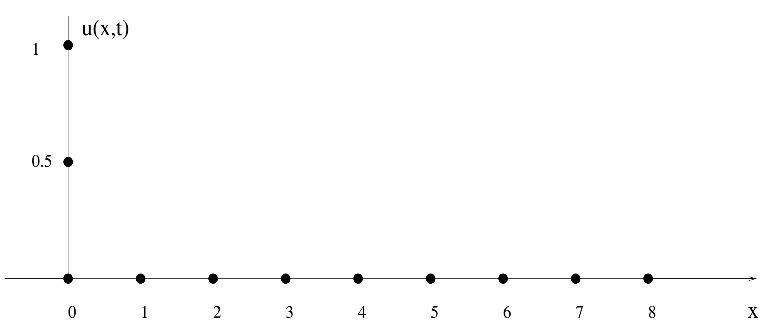
(i) $t = 0$



(ii) $t = 1$



(iii) $t = 2$



(iv) $t = 4$

Problem 4 (20 points)

(a) [5] Find the Fourier cosine coefficients for the function $f(x) = x/\pi$ on $0 < x < \pi$.

(b) [15] Solve the steady heat conduction problem in the region of the half plane outside a semicircle:

$$\frac{\partial^2 u}{\partial^2 r} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad r > 1,\ 0 \le \theta \le \pi,$$

$$u_\theta(r,0) = u_\theta(r,\pi) = 0, \qquad u(1,\theta) = \frac{\theta}{\pi}, \quad u \text{ bounded as } r \to \infty.$$

Extra page (Problem 4)

Problem 5 (20 points)

(a) [10] Consider the eigenvalue problem

$$X'' + \lambda^2 X = 0, \qquad X'(0) = 0, \quad X'(1) = -X(1).$$

Find the eigenfunctions $X_n(x)$ and show that the eigenvalues λ_n satisfy the equation $\lambda \tan \lambda = 1$. Show graphically that there are infinitely many eigenvalues and find their approximate value as $n \to \infty$.

(b) [10] Solve the heat conduction problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \ t > 0$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t), \qquad u(x,0) = 1,$$

giving your answer in terms of the eigenvalues λ_n.

You may use without proof the result that a piecewise continuous function $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \quad \textit{where} \quad c_n = \frac{\int_0^1 f(x) X_n(x)\, dx}{\int_0^1 X_n(x)^2\, dx}.$$

Extra page (Problem 5)