THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations - December 2011

MATHEMATICS 300

TIME: 3 hours

NO AIDS ARE PERMITTED.

All seven questions are of equal value. Each question is worth 10 points. A passing mark is 28/70. If you obtain a mark of N/70 it will be treated as a mark of N/60 with a maximum score possible of 60/60.

Value

1. (a) Sketch $S = \{z \in \mathbb{C} : |z + \frac{1}{2}| > \frac{1}{2} \text{ and } |z+1| < 1\}$.

   (b) Find and sketch the image of S under the mapping $w = f(z) = 1 + \dfrac{1}{1+z}$.

2. (a) Carefully state the Cauchy-Riemann equations and explain their connection with the differentiability and analyticity of a function $f(z)$.

   (b) Find all entire functions $f = u + iv$ for which $v = u^2$.

3. Let $f(z) = \dfrac{z^2}{z^2 + 4z + 3}$.

   (a) Find and classify all singular points of $f(z)$.

   (b) Determine the residues of $f(z)$ at each of its singular points.

   (c) Evaluate $\oint_C f(z)\,dz$ where C is the positively oriented square with corners at $\pm i$ and $-2 \pm i$.

   (d) Where does the Laurent series of $f(z)$ about $z = -1$ converge if it is

      (i) valid near $z = -1$?

      (ii) valid for large $|z|$?

   (e) Find the first four nonzero terms of the Laurent series of $f(z)$ about $z = -1$ which is valid near $z = -1$.

4. (a) Show that the series $\sum_{n=0}^{\infty} e^{-nz}$ converges uniformly in any half plane $\text{Re}\, z \geq \delta$, for any fixed $\delta > 0$.

   (b) Evaluate $f(z) = \sum_{n=0}^{\infty} e^{-nz}$ for $\text{Re}\, z > 0$.

   (c) Evaluate $\sum_{n=1}^{\infty} n^2 e^{-n}$. Be sure to justify your steps.

5. Consider the improper integral $I(m) = \int_0^\infty \frac{x^m}{1+x^8} dx$, where the constant m is an integer, $m = 0, \pm 1, \pm 2, \ldots$.

(a) For which values of m does $I(m)$ converge?

(b) Use contour integration to evaluate $I(m)$, justifying your calculations.

6. Suppose γ_1 and γ_2 are arcs in the z-plane that intersect at z_0. Let α be the angle of intersection. Let $w = f(z)$. In the w-plane, find the angle of intersection at $w_0 = f(z_0)$ of the image arcs $f(\gamma_1)$ and $f(\gamma_2)$ when

(a) $f(z) = z$;

(b) $f(z) = 1 + z$;

(c) $f(z) = 1 + z^3$;

(d) $f(z) = 1 + \bar{z}$;

(e) $f(z) = 1 + \bar{z}^3$.

[Note that the angle of intersection in the w-plane could depend on the value of z_0.]

7. (a) Suppose $\operatorname{Res}_{z=0} f(z) = A$. Let α be a complex constant. Evaluate $\operatorname{Res}_{z=0} f(\alpha z)$.

(b) Evaluate $\oint_{|z|=2} \frac{z^m}{1+z^3} dz$ where the constant m is an integer, $m = 0, \pm 1, \pm 2, \ldots$.