Final Exam

December 11, 2014
12:00–14:30

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (5 points)
Express

$$(1-i)^{10}$$

in the form $a+ib$, for $a, b \in \mathbb{R}$.

Problem 2. (5 points)
On the Riemann sphere, consider the rotation counterclockwise by an angle of $90°$, around the axis through the points 1 and -1. This rotation corresponds to a transformation $z \mapsto w$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$. Find a formula for $w(z)$. Express your answer in the form

$$w = \frac{az+b}{cz+d}, \qquad \text{where } a, b, c, d \in \mathbb{R}\,.$$

Problem 3. (5 points)
Use $\text{Log}_{2\pi}$ to define a branch of the multivalued function

$$w = \sqrt{z^2 - 25}\,.$$

(a) Describe the region in which this branch is analytic.
(b) Evaluate this branch of w at $z = 0$. Give your answer in the form $a+ib$, with $a, b \in \mathbb{R}$, and simplify it as much as possible.

Problem 4. (5 points)
(a) Find the imaginary part $v(x+iy)$ of an analytic function $f(x+iy)$, whose real part is given by

$$u(x+iy) = \cos(x)(e^y + e^{-y})\,.$$

(b) Express the function $f = u + iv$ as a function of z.

Problem 5. (5 points)
Find the integral

$$\int_\Gamma (\bar{z}^2 + 3z^2)\, dz\,,$$

where Γ is the semi-circle in the upper half plane starting at the point $z = 2$, and ending at the point $z = 0$. (This circle has its centre at $z = 1$.) Simplify your answer and write it in the form $a + ib$, with $a, b \in \mathbb{R}$.

Problem 6. (5 points)
Find the integral

$$\int_\Gamma z \sin z\, dz\,,$$

where Γ is the straight line from the point $z = i$ to the point $z = -\pi$. Simplify your answer and write it in the form $a + ib$, with $a, b \in \mathbb{R}$.

Problem 7. (5 points)
Find the integral

$$\oint_\Gamma \frac{\sin(3z)\, dz}{z^2(z-1)^3}\,,$$

where Γ is the circle of radius 2, centred at the origin, traversed once in the counterclockwise direction. Simplify your answer and write it in the form $a + ib$, with $a, b \in \mathbb{R}$.

Problem 8. (5 points)
Find the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 - x + 1}\,.$$

Simplify your answer as much as possible.

Problem 9. (5 points)
Find the Laurent expansion valid in the region $1 < |z-3| < 2$ of the meromorphic function

$$f(z) = \frac{1}{(z-1)(z-3-i)}.$$

This means finding $a_n \in \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-3)^n, \qquad \text{for all } 1 < |z-3| < 2.$$

Simplify your formula for a_n as much as possible.

Problem 10. (5 points)
Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2+i)^n}{n^2} z^n.$$

Problem 11. (10 points)
True or false? Write your answers in your exam booklet. No justification necessary.

(a) The function $f(z) = \frac{z}{\sin(z)}$ has a removable singularity at the origin.

(b) The function $f(z) = e^z$ has an essential singularity at $z = \infty$.

(c) The function $f(z) = \frac{1}{\sin(1/z)}$ has an essential singularity at the origin.

(d) Every function which is analytic in a domain D, is the complex derivative of another function which is analytic in D.

(e) Suppose f is a meromorphic function in $\mathbb{C}$, whose zeroes and poles are as follows: a zero of order 5 at $z = i$, a pole of order 3 at the origin, and a pole of order 7 at $z = 3$. Let γ be the circle of radius 2 centred at the origin, traversed once in counterclockwise direction. Then $f \circ \gamma$ is a closed contour in $\mathbb{C}$, which winds around the origin precisely twice, counterclockwise (in total).

(f) $\operatorname{Log}(1/z) = -\operatorname{Log}(z)$, for all z which are not real.

(g) If Γ is a simple closed curve in $\mathbb{C}$, and z_0 a point in the interior of the region surrounded by Γ, then for every entire function f it is true that $f(z_0)$ is equal to the average of $f(z)$ over Γ.

(h) Suppose that f is analytic at a point $z \in \mathbb{C}$, with $f(z) \neq 0$. Then $\dfrac{f(\zeta)}{\zeta - z}$ considered as a function in ζ, with z fixed, has a simple pole at z with residue $f(z)$.

(i) Suppose that Γ is a simple path in $\mathbb{C}$. Then the formula $f(z) = \displaystyle\int_\Gamma \frac{d\zeta}{\zeta - z}$ describes a function which is analytic everywhere in $\mathbb{C}$, except on the curve Γ.

(j) If the the Taylor expansion of the analytic function $f(z)$ at the origin has radius of convergence 9, then the Taylor expansion of $f(z^2)$ at the origin has radius of convergence 3.