Be sure this exam has 9 pages including the cover

The University of British Columbia

Sessional Exams – 2010 Term 2
Mathematics 303 Introduction to Stochastic Processes
Dr. G. Slade

Last Name: _____ **First Name:** _____

Student Number: _____

This exam consists of **8** questions worth **10** marks each. No aids are permitted.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

(10 points)   1. You are managing a vending machine which generates an average daily profit of $2000 when it is working, but it breaks down each day independently with probability $q = 0.2$. A repair company offers a service contract to restore the machine to working order with probability p in one day whenever it breaks down, at a cost of $\$\frac{100}{1-p}$ each day (whether the machine is working or not). You get to choose p. What value of p should you choose in order to maximize your expected profit?

2. The two-parameter Ehrenfest urn model is defined as follows. A total of M balls are divided between two urns, urn A and urn B. A ball is chosen uniformly at random. If it is chosen from urn A then it is placed in urn B with probability b, and otherwise it is returned to urn A. Similarly, if it is chosen from urn B then it is placed in urn A with probability a, and otherwise it is returned to urn B. Let X_n be the number of balls in urn A at time n.

(3 points)   (a) Determine the transition probabilities for this Markov chain.

(4 points)   (b) Guess (or otherwise determine) the stationary distribution for this Markov chain.

(3 points)   (c) Verify that your guess is correct and that the Markov chain is reversible.

3. Smith owns two umbrellas which he uses on trips between his house and the casino. On each (one way) trip, he takes an umbrella with him if it is raining and if there is an umbrella at his departure location. He never takes an umbrella if it is not raining. It rains on each trip independently with probability p. Let X_n denote the number of umbrellas at his current location.

(4 points)   (a) Find the transition matrix for this Markov chain.

(6 points)   (b) Find the probability that he gets wet (i.e., it is raining and both umbrellas are at the other location).

(10 points) 4. Consider the branching process in which an individual has either 0 or 3 offspring, each with probability $\frac{1}{2}$. Initially there is one individual. Compute the probability that the branching process survives forever.

5. Vehicles pass a point on the road according to a Poisson process with rate 1 per minute. Suppose that on average 20% of the vehicles are trucks, and the rest are cars.

(3 points)   (a) What is the probability that at least two trucks pass in a 5 minute period?

(3 points)   (b) Given that 2 trucks have passed in a 5 minute period, what is the expected number of vehicles that have passed in that time?

(4 points)   (c) If ten vehicles have passed in a 5 minute period, what is the probability that two of them were trucks?

6. A customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time required for service by server i is an exponential random variable with rate μ_i, for $i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3.

(3 points)   (a) Find the probability that server 3 will still be busy when you move over to server 2.

(3 points)   (b) Find the probability that server 3 will still be busy when you move over to server 3.

(4 points)   (c) Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the service in progress to end before you can enter service.)

(5 points)   7.   (a) Recall that, in the $M/M/1$ queueing model, customers arrive according to a Poisson process with rate λ and a single server provides service at rate μ. The state of the system at time t is the total number of customers present (the one being served plus those waiting in the queue for service). Using any method, show that the limiting probabilities are given, when $\lambda < \mu$, by $P_j = (\lambda/\mu)^j(1 - \lambda/\mu)$ for $j = 0, 1, \ldots$.

(5 points)   (b) Determine the average number of customers present.

8. Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

(7 points)   (a) What fraction of the attendant's time will be spent servicing cars?

(3 points)   (b) What fraction of potential customers are lost?