# The University of British Columbia 

Final Examination - April 2014
Mathematics 303
Dr. D. Brydges
Time: 2.5 hours
Last Name
First $\qquad$ Signature

## Student Number

## Special Instructions:

Closed book exam, no calculators. Brief explanation is required whenever it is not clear how answers are obtained. The test has 7 questions and is out of 70 . Be sure this exam has 15 pages including the cover.

## Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
(a) speaking or communicating with other candidates, unless otherwise authorized;
(b) purposely exposing written papers to the view of other candidates or imaging devices;
(c) purposely viewing the written papers of other candidates;
(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)-(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

| Question |  | Points | Score |
| :---: | :---: | :---: | :---: |
|  | 1 | 10 |  |
|  | 2 | 10 |  |
|  | 3 | 10 |  |
|  | 4 | 10 |  |
|  | 5 | 10 |  |
|  | 6 | 10 |  |
|  | 7 | 10 |  |
| Total: |  | 70 |  |

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$\qquad$

1. Consider the Markov chain $\left(X_{n}, n=0,1, \ldots\right)$ whose states and transitions of positive probability are indicated by the oriented arrows in the diagram. One arrow has both orientations; the others only permit transitions in one direction.

(a) (2 points) What are the communicating classes?

Solution: $\{1,2\},\{3,4,5,6,7\}$.
(b) (1 point) Which states, if any, have period 2?

Solution: $\{1,2\}$
(c) (1 point) Which states, if any, have period 3?

Solution: None.
(d) (1 point) Which states, if any, are transient?

Solution: $\{1,2\}$.
(e) (3 points) What are the hypotheses in the main theorem about when the long term probability $\lim _{n \rightarrow \infty} \mathbb{P}_{i}\left(X_{n}=j\right)$ exists and does not depend on $i$.

Solution: The Markov chain $\left(X_{n}, n=0,1, \ldots\right)$ is irreducible, aperiodic and positively recurrent.
(f) (1 point) Do the hypotheses you stated in part (e) hold for $\left(X_{n}, n=0,1, \ldots\right)$ ?

Solution: No. Not irreducible.
(g) (1 point) Explain why $\lim _{n \rightarrow \infty} \mathbb{P}_{i}\left(X_{n}=j\right)$ exists and discuss whether it depends on the initial state $i$.

Solution: It exists because we can apply the main theorem to the Markov chain restricted to the communicating class $\{3,4,5,6,7\}$. When $i=1,2$ the limit is zero because these states are transient.
2. Let $\left(X_{n}, n=0,1, \ldots\right)$ be a Markov chain with transition matrix $P_{i j}$.
(a) (2 points) Let $\pi=\left(\pi_{i}\right)$ be a stationary distribution. What equation expresses the fact that $\pi$ is stationary?

Solution: $\sum_{i} \pi_{i} P_{i j}=\pi_{j}$ for all states $j$.
(b) (2 points) Let $x=\left(x_{i}\right)$ be a probability distribution that satisfies the equations of detailed balance. What are these equations?

Solution: $x_{i} P_{i j}=x_{j} P_{j i}$ for all states $i, j$.
(c) (1 point) Prove that a probability distribution $x=\left(x_{i}\right)$ that solves the equations of detailed balance is stationary.

Solution: $\sum_{i} x_{i} P_{i j}=\sum_{i} x_{j} P_{j i}=x_{j}$.
(d) (5 points) For the Markov chain with states $1,2, \ldots, N$ with transition diagram as shown, find the stationary distribution $\pi=\left(\pi_{i}\right)$ explicitly.


1/3

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Solution: The equations of detailed balance in the case of this problem are $\pi_{i} \frac{2}{3}=\pi_{i+1} \frac{1}{3}$ for $i=1, \ldots, N-1$. This simplifies to $\pi_{i+1}=2 \pi_{i}$. Therefore $\pi_{i}=2^{i-1} \pi_{1}$. $\pi_{1}$ must be chosen so that $\pi_{1}+2 \pi_{1}+\cdots+2^{N-1} \pi_{1}=1$. This is the same as $\pi_{1} \frac{1-2^{N}}{1-2}$. Therefore $\pi_{1}=\frac{1}{2^{N}-1}$ and $\pi_{i}=\frac{2^{i-1}}{2^{N}-1}$.
It is possible but harder to do this problem by solving $\pi P=\pi$ by looking for a solution of the form $\pi_{i}=\lambda^{i}$ as in gamblers ruin.
3. Let $S_{n}$ be the position at time $n$ of symmetric simple random walk on $\mathbb{Z}$ with $S_{0}=0$. Recall that $S_{n}=X_{1}+\cdots+X_{n}$ where $X_{1}, \ldots, X_{n}$ are independent steps which have probability $\frac{1}{2}$ to be 1 and -1 . Let $N^{(99)}$ be the number of visits the random walk makes to the origin up to and including time 99, that is $N^{(99)}=\#\left\{n=0, \ldots, 99 \mid S_{n}=0\right\}$.
(a) (1 point) Write $N^{(99)}$ in terms of the indicator functions $I_{S_{0}=0}, I_{S_{1}=0}, \ldots, I_{S_{99}=0}$.

Solution: $N^{(99)}=\sum_{n=0}^{99} I_{S_{n}=0}$.
(b) (1 point) Express $\mathbb{E} N^{(99)}$ in terms of $\mathbb{P}\left(S_{n}=0\right)$ and show why your answer is correct.

## Solution:

$$
\begin{gather*}
\mathbb{E} N^{(99)} \stackrel{\text { a/ }}{=} \mathbb{E} \sum_{n=0}^{99} I_{S_{n}=0}  \tag{1}\\
=\sum_{n=0}^{99} \mathbb{E} I_{S_{n}=0}=\sum_{n=0}^{99} \mathbb{P}\left(S_{n}=0\right) . \tag{2}
\end{gather*}
$$

(c) (2 points) Find an explicit formula for $\mathbb{E} e^{i k X_{1}}$.

Solution: $\mathbb{E} e^{i k X_{1}}=e^{i k} \mathbb{P}\left(X_{1}=1\right)+e^{-i k} \mathbb{P}\left(X_{1}=-1\right)=\frac{1}{2}\left(e^{i k}+e^{-i k}\right)$.
(d) (2 points) Let $r(k)=\mathbb{E} e^{i k X_{1}}$. Express $\mathbb{E} e^{i k S_{n}}$ in terms of $r(k)$.

## Solution:

$$
\mathbb{E} e^{i k S_{n}}=\mathbb{E}\left(e^{i k X_{1}} \ldots e^{i k X_{n}}\right)=\mathbb{E}\left(e^{i k X_{1}}\right) \ldots \mathbb{E}\left(e^{i k X_{n}}\right)=(r(k))^{n} .
$$

(e) (2 points) What is $\int_{-\pi}^{\pi} e^{i k S_{n}} d k$ when $S_{n}=0$ and what is it when $S_{n}=s$ where $s$ is a nonzero integer? Hint. $e^{i k s}=\cos k s+i \sin k s$.

Solution: For $k=0, \int_{-\pi}^{\pi} e^{i k S_{n}} d k=\int_{-\pi}^{\pi} 1 d k=2 \pi$. For $S_{n}=s$ it is zero: $\int_{-\pi}^{\pi} e^{i k s} d k=\frac{1}{s}[\sin (k s)-i \cos (k s)]_{k=-\pi}^{k=\pi}=0$ because $\sin s \pi=\sin s(-\pi)$ and likewise cos.
(f) (2 points) Show that $\mathbb{E} N^{(99)}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{1-r^{100}}{1-r} d k$ where $r=r(k)$ and $r(k)$ was defined in part (d).

## Solution:

$$
\begin{aligned}
& \mathbb{E} N^{(99)} \stackrel{\text { b) }}{=} \sum_{n=0}^{99} \mathbb{E} I_{S_{n}=0} \stackrel{\text { ed }}{=} \sum_{n=0}^{99} \mathbb{E}\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{i k S_{n}} d k\right) \\
& =\sum_{n=0}^{99} \frac{1}{2 \pi} \int_{-\pi}^{\pi} \mathbb{E}\left(e^{i k S_{n}}\right) d k \stackrel{\text { dd }}{=} \sum_{n=0}^{99} \frac{1}{2 \pi} \int_{-\pi}^{\pi} r^{n} d k \\
& \quad=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{n=0}^{99} r^{n} d k=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{1-r^{100}}{1-r} d k
\end{aligned}
$$

4. A computer virus spreads by forcing an infected computer to transmit copies of the virus to other computers. Let $X_{n}$ be the number of infected computers at time $n$. Assume that $X_{0}=1$ and that $\left(X_{n}, n=0,1, \ldots\right)$ is a branching process where each infected computer independently infects a random number $Z$ of other computers. The distribution for $Z$ is given by $\mathbb{P}(Z=n)=p^{n}(1-p)$ where $n=0,1, \ldots$, and $p$ is a parameter between 0 and 1. Notice that $\sum_{n=0}^{\infty}(1-p) p^{n}=(1-p) \sum_{n=0}^{\infty} p^{n}=(1-p) \frac{1}{1-p}=1$.
(a) (4 points) Give the definition of the generating function $G_{1}(s)$ of $X_{1}$, which is an infinite sum in powers of $s$, and write out the first three terms in this sum.

$$
\text { Solution: } G_{1}(s)=\sum_{x=0}^{\infty} s^{x} \mathbb{P}\left(X_{1}=x\right)=(1-p)+(1-p) p s+(1-p)(p s)^{2}+\ldots
$$

(b) (2 points) Simplify your answer in part (a). The sum can be evaluated.

$$
\text { Solution: } G_{1}(s)=(1-p)\left(1+p s+(p s)^{2}+\ldots\right)=(1-p) \frac{1}{1-p s} \text { for } p s<1
$$

(c) (4 points) Find the probability as a function of $p$, that the virus goes extinct. If you could not do earlier parts explain how to find the extinction probability.

Solution: Must find smallest nonnegative solution $\eta$ to $G_{1}(s)=s$. From part (b), $(1-p) \frac{1}{1-p s}=s, p s^{2}-s+1-p=0, s=1, \frac{1-p}{p}$ so $\eta=\min \left(1, \frac{1-p}{p}\right)$.
$\qquad$
5. Let $Q_{i j}$ be the transition matrix for simple random walk on $\mathbb{Z}$ which steps to the right with probability $\frac{2}{3}$ and to the left with probability $\frac{1}{3}$. Recall that the Metropolis-Hastings algorithm creates a Markov chain $\left(X_{n}\right)$ on some subset of $\mathbb{Z}$ by proposing a transition from $i$ to $j$ with probability $Q_{i j}$ and accepting the proposal if $\frac{b_{j} Q_{j i}}{b_{i} Q_{i j}} \geq U$ where $U$ is distributed uniformly in $[0,1]$. Let $P_{i j}$ be the transition matrix for $\left(X_{n}\right)$.
(a) (3 points) What is the simplest choice of $b=\left(b_{i}\right)_{i \in \mathbb{Z}}$ so that $\left(X_{n}\right)$ has long term probabilities

$$
\pi_{i}=\frac{1}{L}
$$

for $i=1,2, \ldots, L$ and is zero for all other states $i$ ? You do not have to justify your answer in this part.

Solution: You can choose $b_{i}=C \pi_{i}$ where $C$ is any constant, so the simple choice is $b_{i}=1$ for $i=1, \ldots, L$ and otherwise $b_{i}=0$. (The fact that you can use any $C$ is the reason why this algorithm works well on problems where the state space is huge).
(b) (2 points) With the choice you made in part (a) what is $\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} X_{j}^{2}$ when $L=3$ ? What theorem is being used to obtain your answer?

Solution: By the ergodic theorem proved in homework with $f(i)=i^{2}$,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} X_{j}^{2}=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n} f\left(X_{j}\right) \\
= & \sum_{i} \pi_{i} f(i)=\frac{1}{3} 1^{2}+\frac{1}{3} 2^{2}+\frac{1}{3} 3^{2}=\frac{14}{3} .
\end{aligned}
$$

(c) (2 points) With the choice you made in part (a) find $b_{1} P_{12}$ explicitly.

$$
\text { Solution: } b_{1} Q_{12}\left(\frac{b_{2} Q_{21}}{b_{1} Q_{12}} \wedge 1\right)=\frac{2}{3}\left(\frac{1}{\frac{3}{2}} \wedge 1\right)=\frac{1}{3}
$$

(d) (2 points) With the choice you made in part (a) find $b_{2} P_{21}$ explicitly

$$
\text { Solution: } b_{2} Q_{21}\left(\frac{b_{1} Q_{12}}{b_{2} Q_{21}} \wedge 1\right)=\frac{1}{3}\left(\frac{2}{\frac{3}{3}} \wedge 1\right)=\frac{1}{3}
$$

(e) (1 point) With the choice you made in part (a) find $P_{22}$ explicitly. Assume $L>2$.

Solution: $P_{23}=Q_{23}\left(\frac{b_{3} Q_{32}}{b_{2} Q_{23}} \wedge 1\right)=\frac{2}{3}\left(\frac{\frac{1}{2}}{\frac{2}{3}} \wedge 1\right)=\frac{1}{3}$ and $P_{21}=\frac{1}{3}$ so $P_{22}=\frac{1}{3}$ so that the row of the probability transition matrix sums to one.
6. (a) (4 points) A Poisson process $(N(t), t \geq 0)$ of rate $\lambda$ is a counting process defined by three properties. What are they? Write them out in detail if you can.

## Solution:

1. $N(0)=0$.
2. For $s_{1}<s_{2} \leq t_{1}<t_{2}, N\left(t_{2}\right)-N\left(t_{1}\right)$ is independent of $N\left(s_{2}\right)-N\left(s_{1}\right)$.
3. For $s, t \geq 0, \mathbb{P}(N(s+t)-N(s)=n)=\frac{(\lambda t)^{n}}{n!} e^{-\lambda t}$.
(b) Local telephone calls arrive at a fundraiser according to a Poisson process with rate $\frac{18}{10}$ per minute. Long distance telephone calls arrive according to an independent Poisson process with rate $\frac{2}{10}$ per minute. In the following parts try to give answers in simple forms like $e^{-2} \times\left(\frac{6}{10}\right)^{6}$ instead of sums and integrals.
i. (2 points) What is the probability that no long distance call arrives between 2 pm and $2: 05 \mathrm{pm}$.

Solution: $\mathbb{P}\left(N_{L}(5)=0\right)=e^{-2 / 10 \times 5}=e^{-1}$.
ii. (1 point) What is the probability that at least two calls arrive between 3 pm and $3: 03 \mathrm{pm}$.

Solution: The sum of the two independent Poisson processes is a Poisson process $(N(t))$ with rate $\frac{18}{10}+\frac{2}{10}=2$ calls per minute. $\mathbb{P}(N(2) \geq 2)=$ $1-\mathbb{P}(N(2)=0)-\mathbb{P}(N(2)=1)=1-e^{-2 \times 3}-\frac{6}{1!} e^{-2 \times 3}=1-7 e^{-6}$.
iii. (2 points) If there are 5 calls in a five minute period, what is the probability that the first of these calls is in the first minute?

Solution: Easiest to calculate probability that the first call is not within the first minute. Recall that conditional on $N(5)=5$ the times are uniform so we want the probability that all five calls are in the time interval $[1,5]$ which is $\left(\frac{4}{5}\right)^{5}$. Therefore the answer is $1-\left(\frac{4}{5}\right)^{5}$.
Alternative solution: rewrite the event that the first call is in the first minute as $\{N(1) \geq 1\}$. Then

$$
\begin{gathered}
\mathbb{P}(N(1) \geq 1 \mid N(5)=5)=1-\mathbb{P}(N(1)=0 \mid N(5)=5) \\
=1-\frac{\mathbb{P}(N(1)=0, N(5)=5)}{\mathbb{P}(N(5)=5)}=1-\frac{\mathbb{P}(N(1)=0, N(5)-N(1)=5)}{\mathbb{P}(N(5)=5)} \\
=1-\frac{\mathbb{P}(N(1)=0) \mathbb{P}(N(5)-N(1)=5)}{\mathbb{P}(N(5)=5)} \\
=1-\frac{\mathbb{P}(N(1)=0) \mathbb{P}(N(4)=5)}{\mathbb{P}(N(5)=5)} \\
=1-\frac{e^{-2 \frac{8^{5}}{5!}} e^{-8}}{\frac{10^{5}}{5!} e^{-10}}=1-\left(\frac{4}{5}\right)^{5} .
\end{gathered}
$$

iv. (1 point) What is the probability that the first two calls are long distance.

Solution: The probability law of the local and long distance process is equivalent to independently randomly classifying the calls of the total process as "long distance" with probability $\frac{\frac{18}{10}}{\frac{18}{10}+\frac{2}{10}}=\frac{9}{10}$ and "local" with probability $\frac{1}{10}$. Therefore the probability that the first two calls are long distance is $\frac{1}{10} \times \frac{1}{10}$.
$\qquad$
7. A small barbershop, operated by a single barber, has room for at most two customers including the one the barber is working on. Customers arrive according to a Poisson process of rate three per hour, but only enter if the shop is empty or has one customer, otherwise they go away. The barber takes an independent exponential random time with mean $\frac{1}{4}$ hour to cut the hair of each customer. Let $X(t)$ be the number of customers in the barber's shop. The possible values of $X(t)$ are $0,1,2$. Let $P(t)=\left(P_{i j}(t)\right)$ be the probability transition function for this birth/death process.
(a) (3 points) How is $P_{i j}(t)$ defined in terms of a conditional probability of $X(t)$ ?

$$
\text { Solution: } P_{i j}(t)=\mathbb{P}(X(t)=j \mid X(0)=i)
$$

(b) (4 points) Find the derivative $P^{\prime}(0)$ of $P(t)$ at $t=0$ as an explicit $3 \times 3$ matrix, or equivalently, find $v_{i}$ and $q_{i j}$ for $i=1,2,3$ and $j \neq i$.

Solution: $P^{\prime}(0)=\left[\begin{array}{ccc}-v_{1} & 3 & 0 \\ 4 & -v_{2} & 3 \\ 0 & 4 & -v_{3}\end{array}\right]$ where $v_{1}, v_{2}, v_{3}$ are such that the rows sum to zero, that is $v_{1}=3, v_{2}=7, v_{3}=4$.
(c) (2 points) If there is one customer in the shop at time zero what is the expected time until the first transition, that is, until there is either no customers or two customers in the shop?

Solution: A continuous time MC in state $i$ jumps for the first time at a random time $T^{(i)}$ with law $\exp \left[v_{i}\right]$ and this has mean $\frac{1}{v_{i}}$. For state $i=1 v_{1}=7$ so the answer is $\frac{1}{7}$.
To see this in more detail: The first transition out of state 1 takes place at a random time $T^{(1)}$ which is the minimum of the exponential(3) time for first arrival and the exponential(4) time for first departure. The minimum of these two independent exponential times is $\exp (3+4)$ and this has mean $\frac{1}{3+4}$ of an hour.
(d) (1 point) After the barber has been working for a long time what is the probability that a customer arrives, but has to leave because the barbershop is full?

Solution: We have to find the stationary distribution $\pi=\left(\pi_{1}, \pi_{2}, \pi_{3}\right)$ and then the answer is $\pi_{2}$. The equations of detailed balance are

$$
\pi_{1} q_{10}=\pi_{0} q_{01}, \quad \pi_{2} q_{21}=\pi_{1} q_{12}, \quad \pi_{3} q_{32}=\pi_{2} q_{23}
$$

From which

$$
\pi_{1}=\pi_{0} \frac{q_{01}}{q_{10}}, \quad \pi_{2}=\pi_{0} \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}}, \quad \pi_{3}=\pi_{0} \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}} \frac{q_{23}}{q_{32}} .
$$

Putting in the numbers from part (b),

$$
\pi_{1}=\pi_{0} r \quad \pi_{2}=\pi_{0} r^{2} \quad \pi_{3}=\pi_{0} r^{3}
$$

where $r=\frac{3}{4}$. Determine $\pi_{0}$ so that $\pi_{0}+\pi_{1}+\pi_{2}=1$,

$$
\pi_{0}\left(1+r+r^{2}\right)=1
$$

Therefore $\lim _{t \rightarrow \infty} \mathbb{P}\left(X_{t}=2\right)=\pi_{2}=\frac{r^{2}}{1+r+r^{2}}=\frac{9}{37}$.

