The University of British Columbia

Final Examination - April 23, 2013

Mathematics 307/201

Closed book examination — Time: 2.5 hours

Last Name ____________________ **First** _____________ **Signature** __________________

Student Number ____________________

Special Instructions:

1. No books, notes, or calculators are allowed. A MATLAB/Octave formula sheet is provided on the last page. 2. Read the questions carefully and make sure you provide all the information that is asked for in the question. 3. Show all your work. Answers without any explanation or without the correct accompanying work could recieve no credit, even if they are correct. 4. Answer the question in the space provided. Continue on the back of the page if necessary.

1		12
2		15
3		15
4		13
5		15
6		15
7		15
Total		100

[12] **1.**

(a) [3 pts] Write down the definition of the matrix norm $\|A\|$ of a matrix A.

Solution:

$$\|A\| = \max_{\mathbf{x}\neq \mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

(b) [3 pts] Write down the definition of the condition number $\mathrm{cond}(A)$ of a matrix A. Why is this a useful concept?

Solution: The definition is $\mathrm{cond}(A) = \|A\|\|A^{-1}\|$. The condition number bounds the relative error in the solution $\mathbf{x}$of the system $A\mathbf{x} = \mathbf{b}$ in terms of the relative error in $\mathbf{b}$. More precisely, if $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} + \Delta\mathbf{x} = \mathbf{b} + \Delta\mathbf{b}$ then $\|\Delta\mathbf{x}\|/\|\mathbf{x}\| \le \mathrm{cond}(\mathbf{A})\|\boldsymbol{\Delta}\mathbf{b}\|/\|\mathbf{b}\|$

(c) [3 pts] If A is a 2×2 matrix with $\|A\| = 2$, is it possible that $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$? Give a reason.

Solution: No. We have $\|\|A\begin{bmatrix}1\\0\end{bmatrix}\| \le \|A\|\|\begin{bmatrix}1\\0\end{bmatrix}\| = 2\cdot 1 = 2$ but $\|\begin{bmatrix}3\\0\end{bmatrix}\| = 3$.

(d) [3 pts] Find the norm and condition number of $\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\left(= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right)$.

Solution: Since $U = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ satisfies $U^{-1} = U^T (= U)$, it is an orthogonal matrix. Thus multiplication by U doesn't change the norm. So $\|\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}\| = \|\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\| = 2$. Also $\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}^{-1} U^{-1} = \begin{bmatrix}1 & 0\\0 & 1/2\end{bmatrix}^{-1} U$ so $\|\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}^{-1}\| = 1$. Therefore $\mathrm{cond}\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix} = 2\cdot 1 = 2$.

[15] **2.**

(a) [2 pts] Write down the definition of a Hermitian matrix.

Solution: A is Hermitian if $A = A^*$, i.e., A equals its conjugate transpose.

(b) [3 pts] TRUE or FALSE: Eigenvectors for distinct eigenvalues are orthogonal for a real symmetric matrix. Justify your answer.

Solution: TRUE. Eigenvectors of Hermitian matrices are real since if $Ax = \lambda x$ then $\lambda = \langle x, Ax\rangle/\|x\|^2 = \langle Ax, x\rangle/\|x\|^2 = \overline{\lambda}$ [It was okay to assume this is known]. Then if $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ with $\lambda_1 \neq \lambda_2$ then $\lambda_1\langle x_2, x_1\rangle = \langle x_2, \lambda_1 x_1\rangle = \langle x_2, Ax_1\rangle = \langle Ax_2, x_1\rangle = \langle \lambda_2 x_2, x_1\rangle = \overline{\lambda}_2\langle x_2, x_1\rangle = \lambda_2\langle x_2, x_1\rangle$. This is only possible if $\langle x_2, x_1\rangle = 0$

(c) [3 pts] TRUE or FALSE: If a square matrix has repeated eigenvalues, then it cannot be diagonalized. Justify your answer.

Solution: FALSE. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has repeated eigenvalues and is diagonal, hence diagonalized by I.

(d) [2 pts] Write down the definition of a stochastic matrix.

Solution: A matrix is stochastic if it has non-negative entries and its columns sum to 1.

(e) [3 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix?

Solution: 1) There is an eigenvalue $\lambda_1 = 1$. 2) The other eigenvalues satisfy $|\lambda_i| \leq 1$, and 3) The eigenvector for the eigenvalue 1 can be chosen to have non-negative entries.

(f) [2 pts] What can you say about the eigenvalues and eigenvectors of a stochastic matrix if all the entries are strictly positive?

Solution: Same as above except 2') The other eigenvalues satisfy $|\lambda_i| < 1$, and 3') The eigenvector for the eigenvalue 1 can be chosen to have positive entries.

[15] **3**. Suppose that A is a real symmetric matrix.

(a) [5 pts] Explain how to find the largest (in absolute value) eigenvalue of A using the power method.

Solution: Choose a random vector $\mathbf{x_0}$, then define $\mathbf{x}_n$ for $n = 1, 2, 3, \ldots$ by $\mathbf{x}_{n+1} = A\mathbf{x}_n/\|A\mathbf{x}_n\|$. This sequence will converge for large n (possibly with sign flips for a negative eigenvalue) provided the largest eigenvalue in absolute value, has absolute value strictly larger than the rest and $\mathbf{x_0}$ has a component in the direction of the corresponding eigenvector. After the sequence has converged to $\mathbf{x}_*$, $\lambda = \langle \mathbf{x}_*, A\mathbf{x}_* \rangle$.

(b) [5 pts] Explain how to find the eigenvalue of A that is closest to 2 using the power method.

Solution: Choose a random vector $\mathbf{x_0}$, then define $\mathbf{x}_n$ for $n = 1, 2, 3, \ldots$ by $\mathbf{x}_{n+1} = (A - 2I)^{-1}\mathbf{x}_n/\|(A - 2I)^{-1}\mathbf{x}_n\|$. After the sequence has converged to $\mathbf{x}_*$, $\lambda = \langle \mathbf{x}_*, A\mathbf{x}_* \rangle$.

(c) [5 pts] Write down the MATLAB/Octave commands that implement the procedure in (b) with N iterations. Assume that A and N have been defined in MATLAB/Octave, and that the size of A is 1000×1000.

Solution:

```
> x = rand(1000,1);
> for n=[1:N]
>  y=(A-2*eye(1000))\x;
>  x = y/norm(y);
> end
> lambda = dot(x, A*x)
```

[13] **4**. Let S be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}2\\-1\\-1\end{bmatrix}$ and $\begin{bmatrix}4\\1\\1\end{bmatrix}$. Given the MATLAB/Octave calculation

```
> rref([1 2 4; 1 -1 1;1 -1 1])
ans =

   1   0   2
   0   1   1
   0   0   0
```

(a) [7 pts] Find the matrix P that projects onto S.

Solution: The calculation shows that the S is spanned by the independent vectors $[1,1,1]^T$ and $[2,-1,-1]^T$. Let $A = \begin{bmatrix}1 & 2\\1 & -1\\1 & -1\end{bmatrix}$. Then $P = A(A^TA)^{-1}A^T$ which after a short calculations (omitted) gives $P = \begin{bmatrix}1 & 0 & 0\\0 & 1/2 & 1/2\\0 & 1/2 & 1/2\end{bmatrix}$.

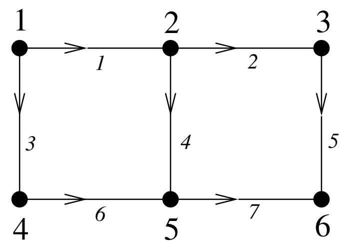
(b) [6 pts] Write down the MATLAB/Octave commands that find the vector in S closest to $\begin{bmatrix}1\\0\\0\end{bmatrix}$

Solution:

```
> P = [1 0 0; 0 0.5 0.5; 0 0.5 0.5]
> x = [1 0 0]'
> P*x
```

(In fact $x \in S$ already so $Px = x$.)

[15] **5**. Consider the following graph, interpreted as a resistor network with all resistances $R = 1$.



(a) [5 pts] Write down the incidence matrix D and the Laplacian matrix L for this graph. Show that (for any graph) $N(L) = N(D)$. Is it true that $R(D^T) = R(L)$? Give a reason.

Solution:

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Since the resistances are all 1, $L = D^T D$. So if $Dv = 0$ then $Lv = D^T Dv = D^T 0 = 0$. On the other hand if $Lv = 0$ then $D^T Dv = 0$ so $\langle v, D^T Dv\rangle = \langle Dv, Dv\rangle = \|Dv\|^2 = 0$ so $Dv = 0$. This shows $N(D) = N(L)$. Then $R(D^T) = N(D)^\perp = N(L)^\perp = R(L^T) = R(L)$.

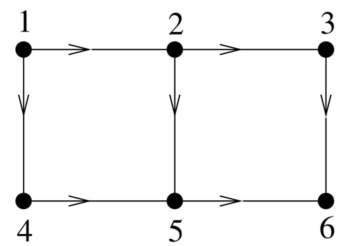
(b) [5 pts] Write down 2 independent loop vectors. Is any other loop vector a linear combination of these? Give a reason.

Solution: Any two of $[1, 0, -1, 1, 0, -1, 0]^T$, $[0, 1, 0, -1, 1, 0, -1]^T$ and $[1, 1, -1, 0, 1, -1, -1]$ are independent loop vectors. Loop vectors form a basis for $N(D^T)$. To compute $\dim(N(D^T))$ start with $\dim(N(D)) = 1$ since the graph is connected. Thus $\dim(R(D)) = 6 - 1 = 5 = \dim(R(D^T))$. Thus $\dim(N(D^T)) = 7 - 5 = 2$. So any two independent vectors in $N(D^T)$ form a basis, and any other loop vector (in fact any other vector in $N(D^T)$) is a linear combination of them.

(c) [5 pts] Suppose that by attaching a battery, the voltage at vertex 1 is held at b_1 and the voltage at vertex 2 is held at b_2. Write down vectors $\mathbf{v}$ and $\mathbf{J}$ so that the equation $L\mathbf{v} = \mathbf{J}$ describes this situation. Explain what each entry of $\mathbf{v}$ and $\mathbf{J}$ represents, and how you can use the entries to compute the effective resistance between vertices 1 and 2.

Solution: $\mathbf{v} = [b_1, b_2, v_3, v_4, v_5, v_6]^T$ and $\mathbf{J} = [c, -c, 0, 0, 0, 0]^T$ where b_1 and b_2 represent the battery voltages at vertices 1 and 2, $v_3, \ldots, v_6$ are the voltages of the remaining vertices, c and $= c$ represent the current flowing in and out of the circuit from the battery, and the remaining entries of $\mathbf{J}$ are zero due to Kirchhoff's law. Using the entries of $\mathbf{v}$ and $\mathbf{J}$ to write the effective resistance, we have $R_e = (b_2 - b_1)/c$.

[15] **6**.
Consider the same graph as in the previous question, now interpreted as an internet where the vertices represent web pages and the arrows represent links.



(a) [5 pts] Write down the stochastic matrix associated with the PageRank algorithm with no damping. Explain what you are doing with vertex 6.

Solution:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1/2 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1/2 & 0 & 1 & 0 & 1/6 \\ 0 & 0 & 1 & 0 & 1 & 1/6 \end{bmatrix}$$

Since there are no outgoing links from vertex 6, we distribute the probability equally over the whole net. Alternatively we could make a single link from vertex 6 to iteself in which case the last column becomes $[0, 0, 0, 0, 0, 1]^T$. In this case the limiting distribution will be concentrated just on vertex 6.

(b) [4 pts]

What is the stochastic matrix associated with the PageRank algorithm with damping factor $\alpha = 1/2$. What happens to the eigenvalues as α tends to 0 (complete damping)?

Solution: Let Q be the matrix with every entry 1/6. Then the new stochastic matrix is $S = (1/2)P + (1/2)Q$. As α tends to zero the eigenvalues of $S_\alpha = \alpha P + (1 - \alpha)Q$ tend to the eigenvalues of Q. But Q is a projection matrix onto a on dimensional space. Thus Q has one eigenvalue of 1 and all the rest zero.

(c) [3 pts]

Starting with equal probabilities for each page, what are the probabilities of being on each page after one step (with $\alpha = 1/2$)? Based on this calculation, which page has the highest rank?

Solution: If $\mathbf{x}_0 = (1/6)[1,1,1,1,1,1]^T$ then $P\mathbf{x}_0 = (1/36)[1,4,4,10,13]^T$ and $Q\mathbf{x} = (1/6)[1,1,1,1,1,1]^T$. Thus $S\mathbf{x_0} = (1/72)[7,10,10,10,16,19]^T$ so page 6 has the highest rank after one step.

(d) [3 pts]

Write down the MATLAB/Octave code that would compute the ranking of each page using the `eig` command.

Solution: Assuming that S has been defined in MATLAB/Octave,

```
> [V,D]=eig(S)
```

computes the eigenvalues and vectors. Assuming that the eigenvalue 1 is the first diagonal entry in the matrix D, the corresponding eigenvector is $V(:,1)$. To get the ranking we must scale so that the sum is one:

```
> V(:,1)/sum((V:,1))
```

[15] **7**. Suppose that the matrix A has the following singular value decomposition:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}.$$

where $\sigma_1 \geq \sigma_2 > 0$.

(a) [5 pts] What are $\|A\|$ and $\operatorname{rank}(A)$

Solution: $\|A\| = \sigma_1$ is the largest singular value. The rank of A is number of (non-zero) singular values. So $\operatorname{rank}(A) = 2$.

(b) [5 pts] Write down an orthonormal basis for $N(A)$ and an orthonormal basis for $R(A)$.

Solution: We pick out the appropriate columns of U and V (where $A = U\Sigma V^T$. The third column of V, namely $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$ is an orthonormal basis for $N(A)$ while the columns of U, namely $[0,1]^T$ and $[1,0]^T$ form a basis for $R(A)$.

(c) [5 pts] What are the eigenvalues and eigenvectors of A^*A and AA^*?

Solution: $A^*A = V\Sigma^T\Sigma V^T = V \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$ so the eigenvalues are σ_1^2, σ_2^2 and 0 and the eigenvectors are columns of V, i.e., $[1/\sqrt{2}, 1/\sqrt{2}, 0]$, $[0,0,1]^T$ and $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$. Similarly, AA^* has eigenvalues σ_1^2 and σ_2^2 with eigenvectors $[0,1]^T$ and $[1,0]^T$.

The End

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable **x** to be 3
`x = [1 2 3]`	set **x** to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set **x** to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of **x** by 3
`x+3`	add 3 to each element of **x**
`x+y`	add **x** and **y** element by element
`A*x`	product of matrix A and column vector **x**
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors **x** and **y**
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector **x**
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of **x** (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of **x** on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of **x**

`A\b`	returns the solution **x** to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector **x**
`vander(x)`	returns the Vandermonde matrix for the points of **x**
`polyval(a,x)`	returns the values of the polynomial $a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n$ at the points of **x**
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector **f** using N points (pads **f** with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of **y** against the points of **x** using blue dots
`plot(x,y,'r-')`	plots the points of **y** against the points of **x** using red lines
`semilogy(x,y,'bo')`	plots **y** against **x** using a logarithmic scale for **y**
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of **z** against the points of **x** and **y** using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands ... for each