The University of British Columbia

Final Examination - April 26, 2014

Mathematics 310

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ______________

Student Number ________ **Section Number** ________ **Instructor** ____________

Special Instructions:

No books, notes, or calculators are allowed.

Senate Policy: Conduct during examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;

(b) purposely exposing written papers to the view of other candidates or imaging devices;

(c) purposely viewing the written papers of other candidates;

(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		18
2		12
3		15
4		12
5		16
6		18
7		9
Total		100

[18] **1**. Label the following statements as TRUE or FALSE. You do not need to justify your answer.

(a) In any vector space V over a field $\mathbb{F}$, $ax = ay$ implies that $x = y$ for any $x, y \in V$ and any $a \in \mathbb{F}$.

Answer:

(b) Let W be the xy-plane in $\mathbb{R}^3$; that is $W = \{(a_1, a_2, 0) \mid a_1, a_2 \in \mathbb{R}\}$. Then W is a subspace of $\mathbb{R}^3$ and W is isomorphic to $\mathbb{R}^2$.

Answer:

(c) There exists a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that $T(0, 0, 1, -1) = (1, 1, 0)$ and $T(0, 0, -3, 3) = (-1, -1, 0)$.

Answer:

(d) Any linear operator on a finite-dimensional vector space over a field $\mathbb{F}$ has a Jordan canonical form.

Answer:

(e) For every linear operator T on a finite-dimensional inner product space V and every ordered basis β for V, we have $[T^*]_\beta = \left([T]_\beta\right)^*$.

Answer:

(f) Let V be a finite-dimensional inner product space V. If T is a normal operator on V and x is an eigenvector of T, then x is also an eigenvector of T^*, where T^* is adjoint of the operator T.

Answer:

[12] **2**. Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be a basis for a vector space V.

(a) Prove that the following set

$$\gamma = \{v_1 + v_2 + \cdots + v_n,\, v_2 + v_3 + \cdots + v_n,\, \cdots,\, v_{n-1} + v_n,\, v_n\}$$

is also a basis for V.

(b) If $x \in V$ and $[x]_\beta = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$, find $[x]_\gamma$.

[15] **3**. Define $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b-2c) + (2b-4c+2d)x + (a-d)x^2,$$

where $a, b, c, d \in \mathbb{R}$. Let

$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

and $\gamma = \{1, x+2, x^2\}$.

(a) Find $[T]_\beta^\gamma$, where $[T]_\beta^\gamma$ is the matrix representation of T in the ordered bases β and γ.

(b) Find a basis for the kernel $N(T)$ of T.

(c) Determine if T is onto.

[12] **4**. Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y + 3z, \quad f_2(x, y, z) = 2y - z, \quad f_3(x, y, z) = 3z,$$

where $(x, y, z) \in \mathbb{R}^3$.

(a) Prove that $\{f_1, f_2, f_3\}$ is a basis for the dual space V^*.

(b) If $\{v_1, v_2, v_3\}$ is the basis for V for which $\{f_1, f_2, f_3\}$ is the dual basis, find the vector v_3.

[16] **5**. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, and let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by $L_A(x) = Ax$ for $x \in \mathbb{R}^3$.

(a) Find a basis for each eigenspace of L_A.

(b) Find a basis for each generalized eigenspace of L_A.

(c) Find a Jordan canonical basis γ for L_A.

(d) Find a matrix Q such that $J = Q^{-1}AQ$, where J is a Jordan canonical form for A.

[18] **6**. $P_2(\mathbb{R})$ is an inner product space with the following inner product:

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear operator defined by

$$T(a+bx+cx^2) = \left(a - \frac{b}{\sqrt{3}}\right) + \left(-a\sqrt{3} + b - \frac{c}{\sqrt{3}}\right)x + cx^2$$

for $a, b, c \in \mathbb{R}$.

(a) Find the positive real number k such that

$$\beta = \left\{\frac{1}{\sqrt{2}},\ x\sqrt{\frac{3}{2}},\ k(3x^2-1)\right\}$$

is an orthonormal basis for the inner product space $P_2(\mathbb{R})$.

(b) Prove that T is self-adjoint.

(c) Find an orthonormal basis γ for $P_2(\mathbb{R})$ consisting of eigenvectors of T.

[9] **7**. Let $T : V \to V$ be a normal operator on a finite-dimensional complex inner-product space V. Assume that $T^7 = T^6$.

(a) Prove that the eigenvalues of T are real.

(b) Prove that T is self-adjoint.

(c) Prove that $T^2 = T$.

The End