Be sure this exam has 12 pages including the cover

The University of British Columbia

Sessional Exams – 2008/2009 Winter Term 2
Mathematics 318 Probability with Physical Applications,
D. Brydges

Name: ______________________________

Student Number: ______________________________

This exam consists of **8** questions worth **10** marks each. No aids other than calculators are permitted.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

Tables on last page.

(2 points) 1. (a) Define the term “event”.

(3 points) (b) What is the definition of independence for two events A, B?

(2 points) (c) Are there events A such that A is independent of A? Don’t guess, reason from definition in part (b).

(3 points) (d) Let A, B, C be independent events. Prove that A and $B \cup C$ are independent.

(4 points) 2. (a) A coin with $\mathbb{P}\{\text{head}\} = p$ is tossed repeatedly. Let N be the first toss on which a head occurs. What is $\mathbb{P}\{N = n\}$ and what is $\mathbb{P}\{N > n\}$?

(4 points) (b) Compute $\mathbb{P}\{N > n + m | N > n\}$ and interpret your answer.

(2 points) (c) A protein makes 10^{15} attempts to fold into a more stable configuration every second. Each attempt is successful with probability 3×10^{-15}. Let T be the time to first success in *seconds*. Find, approximately, as a function of t, the probability that $T \leq t$.

(10 points) 3. A manufacturer of car batteries finds that a sample of five batteries have lifetimes 1.9, 2.4, 3.0, 3.5, 4.2 years, and on this basis, decides to claim that his batteries will last, on the average, three years with a standard deviation of six months. Is this reasonable? Assume the battery lifetimes are normally distributed.

4. An apartment block has two lights, one on the left and the other on the right of the entrance. Each light has one lightbulb. These lightbulbs are replaced immediately upon failure and each lightbulb has a random lifetime, independent of other lightbulbs, which has exponential distribution with mean lifetime equal to one month.

(1 points) (a) What is the probability density for the lifetime (in months) for a bulb?

(2 points) (b) Neither lightbulb has been replaced for a month. What is the probability that the left one will last for at least another month?

(3 points) (c) What is the probability at least one will fail in the next month?

(2 points) (d) What is the probability that in the left light exactly four bulbs will have to be replaced in the next two months?

(2 points) (e) What is the probability that exactly n bulbs have to be replaced in the left light before any bulb is replaced in the right light? Hint: use conditioning.

(2 points) 5. (a) Define the moment generating function $M(t)$ for a random variable X

(2 points) (b) Express $\mathbb{E}X$ in terms of $M(t)$

(4 points) (c) Compute $M(t)$ for the Poisson random variable with parameter λ.

(2 points) (d) Prove that when X and Y are independent Poisson(λ) then $X + Y$ is Poisson(2λ). You may assume that the moment generating function determines the law.

6. Recall that in gamblers ruin Smith repeatedly plays a game and in each game with probability p he wins one dollar from the bank and with probability q he loses one dollar to the bank. He plays until he is broke or until he wins all the bank's money. Smith starts with $\$i$ and the bank has $\$N - i$. Let p_i be the probability that Smith goes broke.

(3 points) (a) Explain why p_i solves the equations $pp_{i+1} + qp_{i-1} = \cdots$ for $i = 1, 2, \dots N - 1$.

(2 points) (b) Two more equations are needed to solve for the unknowns p_i, $i = 0, \dots, N$. What are they?

(3 points) (c) Find p_i. Hint: $1, \lambda^i$.

(2 points) (d) Suppose, before each game, Smith dies of a heart attack with probability ϵ, independent of everything else. If he dies then the probability that he subsequently wins or goes broke is assumed to be zero. What equations for p_i replace the ones in part (a)?

7. Let $(X_n,\ n = 0, 1, 2, \dots)$ be a Markov chain with stationary probability distribution π.

(2 points) (a) What equation expresses the fact that π is a stationary probability distribution for a Markov chain with transition matrix P?

(2 points) (b) If $\mathbb{P}\{X_n = i\} = \pi_i$, what is $\mathbb{P}\{X_{n+1} = i\}$? Derive the equation you stated in part (a).

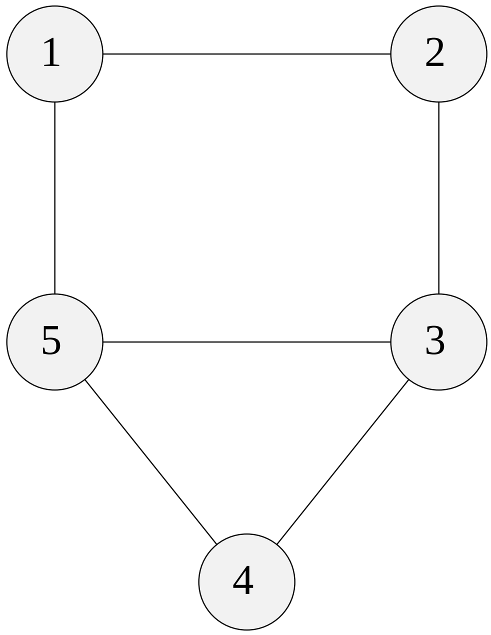
(2 points) (c) If the Markov chain is reversible, what equations does π solve which express this reversibility?

(2 points) (d) Show that the equations you stated in the last part imply the equations of part (a).

(2 points) (e) Given a deck of cards in some order, put them in a new order by picking a card at random from the deck and placing it at the top. This is a Markov chain whose states are the possible orders of the cards in the deck. Is it a reversible Markov chain? Explain briefly.

8. Consider the Markov chain with the transition diagram in the picture. The lines signify the nonzero probability transitions and, given a state, all transitions out of that state have equal probability.

(4 points) (a) What does it mean to say that a Markov Chain is irreducible, positively recurrent and aperiodic. Which of these properties hold for this Markov chain?



(2 points) (b) Mark on the diagram the proportion of time the Markov chain spends in each state

(2 points) (c) Starting in state 4 what is the expected time of first return?

(2 points) (d) State and verify the equations of detailed balance for states $1, 5$.

Table 1: Mean and Variances

Distribution	Mean	Variance
Bin (n, p)	np	$np(1-p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Table 2: cdf of normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

PERCENTAGE POINTS, CHI-SQUARE DISTRIBUTION (Continued)

$$F(\chi^2) = \int_0^{\chi^2} \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n-2}{2}} e^{-\frac{x}{2}}\,dx$$

n \ F	.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

PERCENTAGE POINTS, STUDENT'S t-DISTRIBUTION

This table gives values of t such that

$$F(t) = \int_{-\infty}^{t} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx$$

for n, the number of degrees of freedom, equal to 1, 2, . . . , 30, 40, 60, 120, ∞; and for $F(t)$ = 0.60, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995, and 0.9995. The t-distribution is symmetrical, so that $F(-t) = 1 - F(t)$.

n \ F	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	2.397	1.860	2.306	2.896	3.355	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291