

The University of British Columbia
Final Examination - December 7, 2012
Mathematics 320

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed. Marks depend on quality of proofs. You can use parts of problems to prove other parts without having done the parts you use.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. Define

(a) (3 points) $\liminf a_n$

(b) (4 points) $\lim_{x \rightarrow a} f(x)$

(c) (3 points) metric

2. Let X, Y be sets and let $f : X \rightarrow Y$. Let A be any subset of X and let B any subset of Y .

(a) (2 points) Define $f^{-1}(B)$.

(b) (2 points) Define $f(A)$.

(c) (6 points) Which of the following statements is true for all X, Y, A, f ? (i) $f^{-1}(f(A)) = A$; (ii) $f^{-1}(f(A)) \subset A$; (iii) $f^{-1}(f(A)) \supset A$. Justify your answer by counterexample and proof.

3. $[0, 1]$ is an interval of the real line. For an integer $n \geq 0$ let D_n be the subset of $[0, 1]$ consisting of all points $d = \frac{a_0}{2^0} + \frac{a_1}{2^1} + \cdots + \frac{a_n}{2^n}$ where a_0, a_1, \dots, a_n are each either 0 or 1. A number d in $[0, 1]$ is said to be *dyadic* if it belongs to D_n for some n . Let D be the set of all dyadic numbers in $[0, 1]$.
- (a) (2 points) Show that D is countable without assuming that the rationals are countable.
- (b) (4 points) For $x \in [0, 1]$ let $E = \{d \in D \mid d \leq x\}$. Prove that $x = \sup E$. Hint: $a_0 = 0, a_1 = 1$ if $x \in [\frac{1}{2}, 1)$, $a_1 = 0$ if $x \in [0, \frac{1}{2}]$.
- (c) (2 points) Starting with the definition of “dense” show that D is a dense subset of $[0, 1]$.
- (d) (2 points) Let $\{d_n\}$ be any sequence of distinct points in D such that every point in D appears as a term in $\{d_n\}$. Does $\{d_n\}$ always have a subsequence that converges to $\frac{1}{2}$?

4. Let $\{a_n\}$ be a sequence of real numbers and let $c_n = \frac{a_1+a_2+\dots+a_n}{n}$.
- (a) (7 points) Assume $\lim a_n = L$, where L is a real number. Starting with the definition of convergence for a sequence prove that $\{c_n\}$ also converges to L . Note: since $\{a_n - L\}$ is a convergent sequence, it is bounded by a constant M .
- (b) (3 points) Find an example of a divergent sequence $\{a_n\}$ for which the sequence $\{c_n\}$ converges. Justify your example.

5. You must justify your conclusions.

(a) (2 points) Define what it means for $\sum a_n$ to be conditionally convergent.

(b) (2 points) Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n+1}\right)^3$ is conditionally convergent, absolutely convergent or divergent.

(c) (4 points) Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{n(n!)^2 2^n}$ is convergent.

(d) (2 points) Can $\sum_{n=1}^{\infty} a_n$ be convergent if $\limsup \left| \frac{a_{n+1}}{a_n} \right| = 2$?

6. The sets in this question are subsets of a metric space X .

(a) (2 points) Define what it means for A and B to be separated sets.

(b) (2 points) If A and B are separated sets and E is another set, show that $A \cap E$ and $B \cap E$ are separated.

(c) (4 points) Let E_1, E_2 be connected sets such that $E_1 \cap E_2 \neq \emptyset$ and let $E = E_1 \cup E_2$. Show that E is connected.

(d) (2 points) Is the interior of a connected set connected? (Proof or counterexample).

7. Let f be a real valued function defined on the whole real line.
- (a) (4 points) State the mean value theorem and use it to show that if $f'(x) = 0$ on an interval (a, b) then f is constant in this interval.
- (b) (2 points) In the previous part can one also conclude that $f(a) = f(b)$? Justify your answer.
- (c) (4 points) if $|f(x) - f(y)| \leq \frac{|x-y|}{|\log|x-y||}$ for all real $x \neq y$, show that f is constant.

8. Let $F \supset E$ be subsets of a metric space X . We say that a real valued continuous function f that is defined on E extends to a continuous function defined on F if there exists a continuous function g defined on F such that $g(x) = f(x)$ for $x \in E$.
- (a) (2 points) Define what it means to say that $f : X \rightarrow R$ is uniformly continuous on $E \subset X$.
- (b) (2 points) Give an example of a real valued function f that is defined and continuous on $(0, 1]$ which cannot be extended to a continuous function on $[0, 1]$.
- (c) (2 points) If $f : (0, 1] \rightarrow R$ extends to a continuous function g defined on $[0, 1]$, is g uniformly continuous on $[0, 1]$?
- (d) (4 points) Let f be uniformly continuous on $(0, 1]$. Sketch a proof that f extends to a continuous function g defined on $[0, 1]$. If you are short of time use good judgment in deciding which details of the proof to include.

