The University of British Columbia

Final Examination - December 7, 2012

Mathematics 320

Time: 2.5 hours

Last Name ____________________ **First** ______________ **Signature** ____________________

Student Number ______________________

Special Instructions:

No books, notes, or calculators are allowed. Marks depend on quality of proofs. You can use parts of problems to prove other parts without having done the parts you use.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. Define

   (a) (3 points) $\liminf a_n$

   (b) (4 points) $\lim_{x\to a} f(x)$

   (c) (3 points) metric

2. Let X, Y be sets and let $f : X \to Y$. Let A be any subset of X and let B any subset of Y.

   (a) (2 points) Define $f^{-1}(B)$.

   (b) (2 points) Define $f(A)$.

   (c) (6 points) Which of the following statements is true for all X, Y, A, f? (i) $f^{-1}\big(f(A)\big) = A$; (ii) $f^{-1}\big(f(A)\big) \subset A$; (iii) $f^{-1}\big(f(A)\big) \supset A$. Justify your answer by counterexample and proof.

3. $[0,1]$ is an interval of the real line. For an integer $n \geq 0$ let D_n be the subset of $[0,1]$ consisting of all points $d = \frac{a_0}{2^0} + \frac{a_1}{2^1} + \cdots + \frac{a_n}{2^n}$ where $a_0, a_1, \ldots, a_n$ are each either 0 or 1. A number d in $[0,1]$ is said to be *dyadic* if it belongs to D_n for some n. Let D be the set of all dyadic numbers in $[0,1]$.
   (a) (2 points) Show that D is countable without assuming that the rationals are countable.

   (b) (4 points) For $x \in [0,1]$ let $E = \{d \in D | d \leq x\}$. Prove that $x = \sup E$. Hint: $a_0 = 0$, $a_1 = 1$ if $x \in [\frac{1}{2}, 1)$, $a_1 = 0$ if $x \in [0, \frac{1}{2}]$.

   (c) (2 points) Starting with the definition of "dense" show that D is a dense subset of $[0,1]$.

   (d) (2 points) Let $\{d_n\}$ be any sequence of distinct points in D such that every point in D appears as a term in $\{d_n\}$. Does $\{d_n\}$ always have a subsequence that converges to $\frac{1}{2}$?

4. Let $\{a_n\}$ be a sequence of real numbers and let $c_n = \frac{a_1+a_2+\ldots+a_n}{n}$.

   (a) (7 points) Assume $\lim a_n = L$, where L is a real number. Starting with the definition of convergence for a sequence prove that $\{c_n\}$ also converges to L. Note: since $\{a_n - L\}$ is a convergent sequence, it is bounded by a constant M.

   (b) (3 points) Find an example of a divergent sequence $\{a_n\}$ for which the sequence $\{c_n\}$ converges. Justify your example.

5. You must justify your conclusions.

(a) (2 points) Define what it means for $\sum a_n$ to be conditionally convergent.

(b) (2 points) Determine whether $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n+1}\right)^3$ is conditionally convergent, absolutely convergent or divergent.

(c) (4 points) Determine whether $\sum_{n=1}^{\infty}\frac{(2n)!}{n(n!)^2 2^n}$ is convergent.

(d) (2 points) Can $\sum_{n=1}^{\infty} a_n$ be convergent if $\limsup\left|\frac{a_{n+1}}{a_n}\right| = 2$?

6. The sets in this question are subsets of a metric space X.

(a) (2 points) Define what it means for A and B to be separated sets.

(b) (2 points) If A and B are separated sets and E is another set, show that $A \cap E$ and $B \cap E$ are separated.

(c) (4 points) Let E_1, E_2 be connected sets such that $E_1 \cap E_2 \neq \varnothing$ and let $E = E_1 \cup E_2$. Show that E is connected.

(d) (2 points) Is the interior of a connected set connected? (Proof or counterexample).

7. Let f be a real valued function defined on the whole real line.
   (a) (4 points) State the mean value theorem and use it to show that if $f'(x) = 0$ on an interval (a, b) then f is constant in this interval.

   (b) (2 points) In the previous part can one also conclude that $f(a) = f(b)$? Justify your answer.

   (c) (4 points) if $|f(x) - f(y)| \leq \frac{|x-y|}{|\log|x-y||}$ for all real $x \neq y$, show that f is constant.

8. Let $F \supset E$ be subsets of a metric space X. We say that a real valued continuous function f that is defined on E extends to a continuous function defined on F if there exists a continuous function g defined on F such that $g(x) = f(x)$ for $x \in E$.

   (a) (2 points) Define what it means to say that $f : X \to R$ is uniformly continuous on $E \subset X$.

   (b) (2 points) Give an example of a real valued function f that is defined and continuous on $(0, 1]$ which cannot be extended to a continuous function on $[0, 1]$.

   (c) (2 points) If $f : (0, 1] \to R$ extends to a continuous function g defined on $[0, 1]$, is g uniformly continuous on $[0, 1]$?

   (d) (4 points) Let f be uniformly continuous on $(0, 1]$. Sketch a proof that f extends to a continuous function g defined on $[0, 1]$. If you are short of time use good judgment in deciding which details of the proof to include.