The University of British Columbia

Final Examination - December 2013

Mathematics 320

Time: 2.5 hours

Last Name ____________________ **First** ______________ **Signature** ____________________

Student Number ______________________

Special Instructions:

No books, notes, or calculators are allowed. Marks depend on quality of proofs. You can use parts of problems to prove other parts without having done the parts you use.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. Let A be a subset of $\mathbb{R}$ which has a lower bound.

   (a) (4 points) What is the definition of $\inf A$?

   (b) (6 points) Let L be the set of all lower bounds for A. This is a subset of $\mathbb{R}$. Using the definitions of inf and sup show that $\sup L = \inf A$.

2. Let X be a connected metric space.

   (a) (4 points) Define what it means for X to be connected.

   (b) (6 points) Prove that X is not countable.

3. Let X be a compact metric space and let K be a closed subset of X.

   (a) (4 points) According to the definition of compactness, what must you prove in order to show that K is compact?

   (b) (6 points) Prove that K is compact. Hint. The complement of a closed set is open.

4. Let $f(x) = \frac{1}{x}$ for $x \in \mathbb{R}$ and $x \neq 0$.

   (a) (4 points) What is the ϵ, δ definition that f is continuous at 1?

   (b) (6 points) Use your answer in part (a) to show that $f(x)$ is continuous at 1.

5. You must justify your conclusions.

(a) (1 point) Give an example of a divergent series $\sum a_n$ such that $\limsup \frac{|a_{n+1}|}{|a_n|} = 1$.

(b) (2 points) Give an example of a convergent series $\sum a_n$ such that $\limsup \frac{|a_{n+1}|}{|a_n|} = 1$.

(c) (2 points) Can $\sum_{n=1}^{\infty} a_n$ be convergent if $\limsup \left| \frac{a_{n+1}}{a_n} \right| = 2$?

(d) (5 points) For the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ find R such that the series converges for $|x| < R$ and diverges for $|x| > R$.

6. Let $E = \mathbb{Q} \cap [0,1]$ be the set of rational numbers in the unit interval. For a function $f : E \to \mathbb{R}$ we say that f has a *continuous extension* g if (i) g is a real valued function defined for all real numbers in $[0,1]$, (ii) $g(x) = f(x)$ whenever x is in E, (iii) g is continuous at every point in $[0,1]$.

   (a) (2 points) Suppose f has two continuous extensions g and h. Prove that $g(x) = h(x)$ for $x \in [0,1]$.

   (b) (4 points) Define what it means for f to be uniformly continuous on E.

(c) (2 points) Suppose f has a continuous extension g. Show that this implies that f is uniformly continuous.

(d) (2 points) Give an example of a function f which is continuous on E and which has no continuous extension.

7. Let f be a continuous function that maps the unit interval $[0, 1]$ in $\mathbb{R}$ to itself. Assume that f has a derivative f' which is defined and continuous on $[0, 1]$ and that $|f'(x)| < 1$ for $x \in [0, 1]$.

   (a) (4 points) State the mean value theorem.

   (b) (6 points) Show that there is a constant $M < 1$ such that for all x, y in $[0, 1]$, $|f(x) - f(y)| \leq M|x - y|$.

8. You must briefly justify your answers to the following questions. Recall that the Taylor expansion of $f(x)$ in powers of $x - \alpha$ is $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(\alpha)(x-\alpha)^k$.

   (a) (2 points) Are all convergent sequences Cauchy sequences?

   (b) (2 points) Are intervals the connected subsets of $\mathbb{Q}$?

   (c) (2 points) Is the interior of a connected set connected?

(d) (2 points) Let f be a real valued function with derivatives of all orders, all of which are defined on the interval $[1, 2]$ in $\mathbb{R}$. Let $P(x)$ be the Taylor polynomial for f of degree $n-1$ in powers of $x-1$. In terms of f what are the derivatives of $P(x)$ at $x = 1$?

(e) (2 points) What are the first three terms of the Taylor expansion in powers of x for the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0? \end{cases}$$