Math 322, Fall Term 2011
Final Exam

Do not turn this page over until instructed

- On the top of each exam booklet write
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Instructions

- You will have 150 minutes for this exam.
- No books, notes or electronic devices.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you use a result from the lectures or the problem sets, quote it properly.
- Good luck!

1. [8+6+6=20pts]

(a) Define the sign of an element in S_n and the alternating group A_n. Determine the sign of $\sigma = (1\,2\,3\,4\,5)$ in S_8 and the sign of a cycle of type (3,4,5) in S_{12}.

(b) For $n > 2$, prove that the centre of S_n is trivial.

(c) Write an element in S_4 of cycle type $(2, 2)$ and determine the number of elements in its conjugacy class.

2. [7+8+5=20pts]

(a) Write down the class equation of a group and use it to show that a finite group of order p^n, p a prime, has non-trivial centre.

(b) Show that any group of order 45 is abelian and any group of order 33 is cyclic.

(c) Classify the possible abelian groups of order 792.

3. [8+8+6=22pts]

(a) Prove that $f(x) = x^4+2x+2$ is irreducible over $\mathbb{Q}$. Define a prime ideal and maximal ideal in a ring R and give an example of a non-zero prime ideal in $\mathbb{Z}[x]$.

(b) Is 13 a prime in $\mathbb{Z}[i]$? Justify your answer. Show that any ideal in $\mathbb{Z}[i]$ contains a positive integer.

(c) If $f : R \to S$ is a ring homomorphism between commutative rings show that $f^{-1}(P)$ is a prime ideal in R whenever P is a prime ideal in S.

4. [10+6+6=22pts]

(a) Show that $f(x) = x^3 + 2x + 2$ is irreducible in $\mathbb{F}_3[x]$ ($\mathbb{F}_3$ is the finite field with three elements) and use this fact to construct a field with 27 elements that contains $\mathbb{F}_3$.

(b) Consider the polynomial $f(x) = (x^2 + 1)(x^2 - 2)$ over $\mathbb{Q}$. Find a field extension of $\mathbb{Q}$ where $f(x)$ splits completely into linear factors. What is the degree of such a field extension?

(c) Let E/F be a field extension. What does it mean to say that E/F is algebraic? Give a necessary and sufficient condition for an element α in E to be algebraic over F. Give an example of a complex number which is not real and is algebraic over $\mathbb{Q}$.

5. State whether the following are true or false with full justification [8x2=16pts]

(a) If τ is a cycle in S_n then τ^2 is also a cycle.

(b) Let $R = \mathbb{Z}[i]$ be the ring of Euclidean integers. In the polynomial ring $R[X]$ the prime elements are the same as irreducible elements.

(c) If a group of order 25 acts on a set X, then there is an element $x \in X$ such that its orbit has 8 elements.

(d) The number of conjugacy classes in S_n is equal to the number of non-isomorphic abelian groups of order p^n, for any prime p.

(e) The group S_6 has a subgroup H of order 360 whose normaliser is S_6.

(f) The polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ is irreducible in $\mathbb{Z}[x]$.

(g) In $\mathbb{Z}[x]$, every prime ideal is maximal.

(h) If P is a p-Sylow subgroup and Q is a q-Sylow subgroup for two distinct primes p and q, and if $x \in P$, then no conjugate of x lies in Q.