THE UNIVERSITY OF BRITISH COLUMBIA
SESSIONAL EXAMINATIONS – APRIL 2013
MATHEMATICS 323

Time: 2 hours 30 minutes

Instructions: You can use the statements we proved in class, or the theorems proved in the textbook, without proof (except the question 4e); but you need to provide complete statements of all the results you quote. Write your name and student number at the top of each booklet you use, and please number the booklets (e.g. "booklet 2 of 5") if you use more than one.

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1. [10 points] Determine whether the following statements are true or false (you have to include proofs/counterexamples):

(a) Let R be an integral domain, F – a free R-module of finite rank, and M – a torsion R-module. Then there is no injective homomorphism from F to M.

(b) Over an arbitrary integral domain, any submodule of a free module is free.

2. [8 points] Recall that for a module M,

$$\mathrm{Ann}(M) = \{r \in R \mid rm = 0 \quad \forall m \in M\},$$

and for an ideal $I \subset R$,

$$\mathrm{Ann}(I) = \{m \in M \mid rm = 0 \quad \forall r \in I\}.$$

Let R be an integral domain, let M be an R-module, and suppose that $\mathrm{Ann}(M) = IJ$, where I and J are co-maximal ideals in R. Prove that $M \simeq M_1 \oplus M_2$, where $M_1 = \mathrm{Ann}(I)$, and $M_2 = \mathrm{Ann}(J)$.

3. [22 points] In each question, factor the given element into irreducibles in the given ring (or show that it is irreducible). (Include complete proofs of irreducibility of the factors). If the factorization is unique, indicate why; if not unique, please give two.

(a) $x^4 - x^2 + 4$ in $\mathbb{F}_5[x]$.

(b) 12 in $\mathbb{Z}[\sqrt{-3}]$.

(c) 12 in $\mathbb{Z}[\frac{\sqrt{-3}+1}{2}]$. Explain the relationship with part (c).

(d) $3x^4 - 6x^3 + 12x^2 - 18x + 6$ in $\mathbb{Q}[x]$, and in $\mathbb{Z}[x]$.

(e) $x^{p-2} + x^{p-3} + \cdots + 1$ in $\mathbb{F}_p[x]$, where p is a prime.

4. [22 points]

(a) Is the ideal $I = \{a + bi \mid a, b \text{ are both even}\}$ a maximal ideal in $\mathbb{Z}[i]$?

(b) Prove that the ideal I from part (a) is principal, and find the generator of I.

(c) Prove that (7) is a prime ideal in $\mathbb{Z}[i]$. Describe the quotient $\mathbb{Z}[i]/(7)$.

(d) Let $p > 2$ be a prime such that the field $\mathbb{F}_p$ contains an element a such that $a^2 = -1$ (in $\mathbb{F}_p$). Prove that then (p) is not a maximal ideal in $\mathbb{Z}[i]$.

(e) Prove that every prime number p that is congruent to 1 mod 4 can be represented as a sum of two squares.

5. [7 points] Let N be the submodule of $\mathbb{Z}^3$ generated by the elements $\langle 1,2,3\rangle$ and $\langle 4,5,6\rangle$. Describe the quotient $\mathbb{Z}^3/N$ (as a $\mathbb{Z}$-module).

6. [12 points] Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be the linear operator defined by $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, ax_1)$ (with respect to the standard basis), where $a \neq 0$ is a complex number.

(a) Find the minimal polynomial of T.
(b) Find the characteristic polynomial of T.
(c) Prove that $\mathbb{C}^4$ with the $\mathbb{C}[x]$-module structure given by T is a direct sum of four 1-dimensional cyclic $\mathbb{C}[x]$-submodules.

Hint: No heavy computation is required in this problem.

7. [7 points] Classify abelian groups of order 600 up to isomorphism.

8. [12 points] Let $T : V \to V$ be a linear operator. The subspace $W \subseteq V$ is called invariant if $T(W) \subseteq W$.

(a) Give an example of a linear operator on a real vector space of dimension greater than 1 that does not have a 1-dimensional invariant subspace.
(b) Prove that any linear operator on a complex finite-dimensional vector space has at least one 1-dimensional invariant subspace.
(c) Let $S(V)$ be the set of conjugacy classes of linear operators on a given complex vector space V that have exactly one 1-dimensional invariant subspace. Prove that $S(V)$ is in bijection with $\mathbb{C}$.