This examination has 13 pages including this cover

The University of British Columbia
Final Examination – 11 Dec 2008

Mathematics 340
Linear Programming

Closed book examination Time: 150 minutes

Name______________________________ **Signature** ______________________________

UBC Student Number________________

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed.

Rules governing examinations

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Total		100

[12] **1.** Study the following problem.

$$\begin{array}{ll} \text{maximize} & f = 5x_1 + x_2 - \ \ x_3 \\ \text{subject to} & 3x_1 + x_2 - \ \ x_3 \le -2 \\ & 3x_1 - x_2 - 2x_3 \le -3 \\ & x_1 \qquad\qquad\qquad \le \ \ 2 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

(a) By comparing the objective function and the constraints, or otherwise, show that this problem cannot be unbounded. [2 marks]

(b) Find a basic feasible solution. [5 marks]

(c) Find all optimal solutions. [5 marks]

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[12] **2.** Consider this LP:

$$\begin{array}{llrl} \max & 7x_1 + 5x_2 + 2x_3 & & \\ \text{subject to} & x_1 + x_2 & & \leq 5 \\ & -x_1 + 2x_2 + x_3 & & \leq 4 \\ & x_1 - x_3 & & \leq 2 \\ & x_1, x_2, x_3 \geq 0 & & \end{array}$$

(a) Use the entries 0, 5, and 9, in some order, to invent a feasible input $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$. [2 marks]

(b) Find the dual linear program. [2 marks]

(c) Find all solutions of the dual LP. Explain how you know there are no others. (*Clue*: Part (a) may help.) [6 marks]

(d) Find all solutions of the given (primal) LP. Explain how you know there are no others. [2 marks]

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[16] **3.** Here is a pair of similar-looking linear programming problems:

$$(P)\ \text{Maximize } f = 2x_1 + 6x_2 + 4x_3 \quad \text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 &\le 4 \\ 3x_1 - x_2 + x_3 &\le 8 \\ 2x_1 + 2x_2 &\le 6 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

$$(\widetilde{P})\ \text{Maximize } f = 2x_1 + 6x_2 + 4x_3 + 6x_4 \quad \text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 + x_4 &\le 4 \\ 3x_1 - x_2 + x_3 + 2x_4 &\le 8 \\ 2x_1 + 2x_2 + x_4 &\le 6 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Let s_1, s_2, s_3 be the slack variables for the three constraints of (P), in the given order.

(a) Find a Basic Feasible Solution for (P), using x_3, s_2, and x_2 as the basic variables. [4 marks]

(b) Prove that the BFS found in part (a) is optimal for (P). [4 marks]

(c) Find the maximum value in $(\widetilde{P})$, and all decision vectors that achieve it. [8 marks]

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[4] **4.** Given a fixed positive integer N, Claude and Rachel play a simple game. Claude secretly chooses an integer from the set $\{1, 2, \ldots, N\}$, and Rachel guesses what he has chosen. If Rachel's guess is correct, Claude pays her 5 Galactic Currency Units. If her guess is not correct, no currency changes hands.

Find the optimal strategies for both players, and the expected payout to Rachel.

Suggestion: An efficient approach is to make smart conjectures about the desired quantities, and then to confirm the correctness of your proposals by applying well-known theorems.

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[8] **5.** The following dictionary has been produced by solving a certain linear program in standard form:

$$\begin{array}{rl} x_1 = & 1 + x_2 - 2x_4 + x_6 \\ x_3 = & 3 - 4x_2 + 3x_4 - 2x_6 \\ x_5 = & 2 + 3x_2 + 2x_4 \\ \hline z = & 15 - x_2 - 3x_4 \end{array}$$

(a) Find the original problem. [6 marks]

(b) The client that brought you this problem wants to see a maximum value of 17. Suppose you can change the number on the right side of exactly one of the constraints in the original problem. Which constraint will you choose to modify, and by how much, in your first attempt to satisfy the client? Explain. (A well-informed first approximation will suffice.) [2 marks]

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[16] **6.** Different choices for k make for different outcomes in the zero-sum matrix game defined by

$$G(k) = \begin{bmatrix} 2 & k & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

(a) Find an equilibrium pair of strategies, and the row player's payoff, when $k = 2$. [3 marks]

(b) Find an equilibrium pair of strategies, and the row player's payoff, when $k = 3$. [8 marks]

(c) Find the largest interval of k-values around $k = 3$ with this property: the column player's optimal strategy is a mixture of the same two pure strategies that he uses when $k = 3$. Find the equilibrium strategies and the game's value as a function of k in this interval. [5 marks]

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[12] **7.** A standard LP and one of its dictionaries are given below. Slack variables are s_1, s_2.

$$(P) \qquad \begin{array}{lr} \text{Maximize } f = 3x_1 + 6x_2 + 4x_3 & \\ \text{subject to} & x_1 + 4x_2 + 2x_3 \le 9 \\ & x_1 + x_2 + 2x_3 \le 6 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

$$\begin{array}{rcl} x_1 &=& 5 - 2x_3 + (1/3)s_1 - (4/3)s_2 \\ x_2 &=& 1 - (1/3)s_1 + (1/3)s_2 \\ \hline f &=& 21 - 2x_3 - s_1 - 2s_2 \end{array}$$

Use the Dual Simplex Method to solve the new problem created by adding the following constraint to (P):

$$x_1 + x_3 \le 3.$$

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[12] **8.** Consider the following problem, in which $\mathbf{b} = (b_1, b_2)$ is not given explicitly:

$$
\begin{array}{lll}
(P) & \text{Maximize} & f = 2x_1 + 3x_2 + x_3 \\
 & \text{subject to} & x_1 - x_2 + 2x_3 + x_4 = b_1 \\
 & & 4x_1 + 2x_2 - x_3 + x_5 = b_2 \\
 & & x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

(a) Find the set of all pairs (b_1, b_2) for which the given problem has an optimal basic solution with x_2 and x_3 as basic variables. Sketch this set on a Cartesian plane with axes labelled "b_1" and "b_2". [8 marks]

(b) Let $V = V(b_1, b_2)$ denote the maximum value in problem (P) as a function of the parameters. Give a simple formula for $V(b_1, b_2)$ that is valid on the set of $\mathbf{b}$-values found in part (a). [3 marks]

(c) Show that the formula from part (c) is *not* valid for all $\mathbf{b} \in \mathbb{R}^2$. (Suggestion: Show that the true value $V(-1, -1)$ is obviously larger than the formula's preduction.) [1 mark]

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[8] **9.** Given a matrix A of size $m \times n$, a matrix B of size $p \times n$, and a vector $\mathbf{c} \in \mathbb{R}^n$, prove that *exactly one of the following statements must be true*:

(a) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{0}$, $B\mathbf{x} = \mathbf{0}$, and $\mathbf{c}^T\mathbf{x} > 0$.

(b) There exist vectors $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^p$ such that $\mathbf{y} \geq \mathbf{0}$ and $A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{c}$.

(In detail, you must show that any choice of A, B, and $\mathbf{c}$ makes at least one of these statements correct, and then explain why there is no choice of A, B, and $\mathbf{c}$ for which both statements are true.)

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