The University of British Columbia
Final Examinations – April 2006

Mathematics 345
Applied Nonlinear Dynamics and Chaos

Instructor: W. Nagata

Time: $2\frac{1}{2}$ hours

A non-programmable calculator, and one $8\frac{1}{2}'' \times 11''$ page of notes may be used.
No other aids are permitted.

[25]   1. Consider the one-dimensional equation for $\phi(t)$ in the phase circle $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$, given by

$$\dot{\phi} = f(\phi, r) = r\sin\phi - \sin^2\phi, \tag{1}$$

where $r \in \mathbf{R}$ is a parameter.

   (a) Observing in (1) that $f(0, r) = 0$ for all r, use linear stability analysis at the fixed point $\phi^* = 0 \pmod{2\pi}$ to verify that the stability changes as r is increased through $r_c = 0$.

   (b) What type of bifurcation (saddle-node, transcritical, or pitchfork) occurs at (ϕ^*, r_c) $= (0\,(\bmod\,2\pi), 0)$? Find the normal form of this bifurcation. (*Hint:* an alternate expression for the vector field, $f(\phi, r) = r\sin\phi - \frac{1}{2} + \frac{1}{2}\cos(2\phi)$, may simplify calculations.)

   (c) Plot the set $\mathcal{Z} = \{(\phi, r) \in S^1 \times \mathbf{R} : f(\phi, r) = 0\}$ (horizonal axis r, vertical axis ϕ). Noting that $\mathcal{Z}$ divides the ϕr-cylinder into a finite number of open regions where $f(\phi, r) > 0$ or $f(\phi, r) < 0$, sketch all the qualitatively different phase portraits in S^1 that are possible for (1), **for $r \geq 0$ only**. (*Hints:* $f(\phi, r) = \sin\phi(r - \sin\phi)$; it is not necessary to sketch the graphs of ϕ versus $\dot{\phi}$; it is not necessary to solve for all fixed points ϕ^* explicitly as functions of r.)

   (d) There are other bifurcations, in addition to the one referred to in part (a). Give all (ϕ^*, r_c) in $S^1 \times \mathbf{R}$ where these additional bifurcations occur, and state the type (saddle-node, transcritical, or pitchfork) of each of them. Normal forms are not required. (*Hint:* look at your plot of the set $\mathcal{Z}$.)

[25]   2. (a) Consider the two-dimensional system for $(x(t), y(t))$ in the phase plane $\mathbf{R}^2$, given by

$$\dot{x} = y, \qquad \dot{y} = x^2 - 4. \tag{2}$$

   i. Find all the fixed points of (2) in the phase plane.
   ii. Use linear stability analysis to classify each fixed point as hyperbolic or non-hyperbolic. If a fixed point is hyperbolic, determine whether it is an attractor, a repeller, or a saddle point. If a fixed point is non-hyperbolic, determine whether the linearization has pure imaginary eigenvalues, a simple zero eigenvalue, or a double zero eigenvalue.

iii. Find a conserved quantity. You may **either** write down a conserved quantity and then verify that it is indeed conserved, **or** write down the conditions a conserved quantity for this system must satisfy and then construct a function that satisfies these conditions.

iv. Sketch the phase portrait of (2). Indicate the (global) stable manifold and the (global) unstable manifold of any saddle point, and indicate homoclinic orbit(s), if any exist.

(b) Consider the two-dimensional system in the phase plane $\mathbf{R}^2$, given by

$$\dot{x} = y, \qquad \dot{y} = x^2 - 4 - \delta y, \qquad (3)$$

where $\delta \in \mathbf{R}$ is a parameter.

i. Let V be the function $V(x,y) = 4x - \frac{1}{3}x^3 + \frac{1}{2}y^2$. Calculate $\dot{V} = \frac{d}{dt}V(x(t), y(t))$ along trajectories of (3), and show that if $\delta > 0$, then V decreases along all trajectories of (3) except fixed points.

ii. Are there any closed orbits of (3) if $\delta > 0$? Explain briefly.

[25] 3. Consider the one-dimensional map $x_{n+1} = f(x_n)$ in the closed interval $[0, 1]$, where f is the tent map of slope 2, given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2 - 2x & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Note that f is continuous, and attains its maximum value at $x = 1/2$.

(a) Sketch the graph of the map, find all fixed points in the closed interval $[0, 1]$, and determine the stability of each of the fixed points (either graphically or by calculation).

(b) Find explicitly the second iterate $f^2(x) = f(f(x))$ of the map, and sketch its graph. (*Hint:* consider $0 \leq x \leq 1/4$, $1/4 \leq x \leq 1/2$, $1/2 \leq x \leq 3/4$ and $3/4 \leq x \leq 1$.)

(c) Find all cycles of f of period 2.

(d) Show that any cycle of f of period N, where N is any positive integer, is unstable.

(e) Show that the map has sensitive dependence on initial conditions.

[25] 4. Consider the two-dimensional system representing a predator-prey interaction, given by

$$\dot{x} = x\left[x(1-x) - y\right], \qquad \dot{y} = y(x-m), \qquad (4)$$

where $x(t) \geq 0$ is the dimensionless population of the prey species, $y(t) \geq 0$ is the dimensionless population of the predator species, and m is a control parameter, with $0 < m < 1$. Let Q denote the quarter-plane $x \geq 0$, $y \geq 0$.

(a) Show that the fixed points of (4) in Q are $(0,0)$, $(1,0)$ and $(m, m - m^2)$, and there are no other fixed points in Q.

2

(b) Use linear stability analysis to classify each fixed point in Q as hyperbolic or non-hyperbolic. If a fixed point is hyperbolic, determine whether it is an attractor, a repeller, or a saddle point. If a fixed point is non-hyperbolic, determine whether the linearization has pure imaginary eigenvalues, a simple zero eigenvalue, or a double zero eigenvalue.

(c) Sketch the nullclines and direction field for (4) in Q.

(d) Let $V(x, y) = x + y$. Show, given any m in the open interval $(0, 1)$, that one can always find c sufficiently large such that $\dot{V} < 0$ along the portion of the line $V(x, y) = c$ that is in Q. (*Hint:* the set $\dot{V} = 0$ divides Q into regions where $\dot{V} > 0$ and where $\dot{V} < 0$; show that for c sufficiently large, the line $x + y = c$ lies in the region where $\dot{V} < 0$.)

(e) Let T be the closed and bounded triangular region given by

$$T = \{\, (x, y) \in \mathbf{R}^2 : 0 \leq x,\, 0 \leq y,\, x + y \leq c \,\},$$

and show that if c is sufficiently large, then any trajectory $(x(t), y(t))$ of (4) that starts with $(x(0), y(0))$ in T at $t = 0$ then must remain in T for all $t \geq 0$. You may use the result of part (d) without having answered it.

(f) Are there any closed orbits of (4) in Q for $0 < m < 1$? The answer may depend on the value of m. Give as much justification as possible, using results from the lectures and the homework. You may use the result of part (e) without having answered it.

Total marks
[100]

The End