Be sure this exam has 9 pages including the cover

The University of British Columbia

MATH 345, Section 201, Instructor Tai-Peng Tsai

Final Exam – April 2012

Family Name ______________________ **Given Name** ______________________

Student Number ______________________ **Signature** ______________________

A non-programmable calculator and one page of notes may be used.

No other aids are permitted.

problem	max	score
1.	10	
2.	15	
3.	15	
4.	10	
5.	15	
6.	20	
7.	15	
total	100	

(10 pt) 1. Consider the following ODE for $x(t)$ in the phase line $\mathbb{R}$,

$$\dot{x} = x + \frac{rx}{1+x^2}. \tag{1}$$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of fixed point x^* versus r, indicating the stable and unstable branches.

Normal forms for one dimensional bifurcation:

1. Saddle-node: $\dot{x} = r + x^2$, $\dot{x} = r - x^2$
2. Transcritical: $\dot{x} = rx - x^2$
3. Pitchfork: $\dot{x} = rx - x^3$, $\dot{x} = rx + x^3$, $\dot{x} = rx + x^3 - x^5$

2. Consider the following ODE for $x(t)$ in the phase line $\mathbb{R}$,

$$\dot{x} = h + rx - x^2. \tag{2}$$

(10 pt) (a) Plot the bifurcation diagram of fixed point x^* versus r for $h < 0$, $h = 0$, and $h > 0$.

(5 pt) (b) Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields, and identify the bifurcation that occur on the boundaries of those regions.

3. Consider the two-dimensional system for $(x(t), y(t))$ in the phase plane $\mathbb{R}^2$ given by

$$\dot{x} = y, \quad \dot{y} = x^2 - 4. \tag{3}$$

It has two fixed points $(2, 0)$ and $(-2, 0)$.

(5 pt) (a) Use linear stability analysis to classify each fixed point.

(5 pt) (b) Find a conserved quantity and verify that it is indeed conserved.

(5 pt) (c) Sketch the phase portrait of (3). Indicate the (global) stable manifold and the (global) unstable manifold of any saddle point. Indicate the homoclinic orbit(s), if any exists.

4. Consider the perturbed system with a small parameter $\delta > 0$

$$\dot{x} = y, \quad \dot{y} = x^2 - 4 - \delta y. \tag{4}$$

(2 pt) (a) Use linear stability analysis to classify each fixed point.

(4 pt) (b) Find a Liapunov function $V(x, y)$ and verify that V decreases along all trajectories of (4) except fixed points.

(4 pt) (c) Are there any closed orbit of (4)? Explain.

5. A Hopf bifurcation occurs at the origin when $\mu = 0$ for the following system

$$\dot{x} = \mu x + y - x^3, \qquad \dot{y} = -x + \mu y + 2y^3. \tag{5}$$

(5 pt) (a) Decide the stability of the origin when μ varies.

(5 pt) (b) For both $\mu < 0$ and $\mu > 0$, with $|\mu| \ll 1$, sketch the nullclines and the flow directions.

(5 pt) (c) By looking at the local phase portrait, determine and explain whether the bifurcation is supercritical or subcritical. You need not construct a trapping or repelling region.

(20 pt) 6. Consider the quadratic map $x_{n+1} = x_n^2 + c$, where c is a real parameter.

(a) Find and classify all the fixed points as a function of c.

(b) Find the values of c at which the fixed points bifurcate.

(c) For which values of c is there is a 2-cycle?

(d) For which values of c is there is a stable 2-cycle?

7. Multiple choice: Circle all the correct answers.

(6 pt) (a) If the linearization of a 2D nonlinear system at a fixed point is a stable star, the local portrait of the nonlinear system can be

1. a node
2. a spiral
3. a saddle
4. a center

(3 pt) (b) Any closed orbit is a limit cycle.

1. True
2. False

(3 pt) (c) Chaos may occur in a 2D flow.

1. True
2. False

(3 pt) (d) When chaos occurs in the Lorentz system, the trajectory for the same initial condition changes every time we compute it.

1. True
2. False