Full Name: ______________________ Signature: ______________________

Student Number: ______________________

Math 361 Final Exam December 2006 2.5 hours.

Instructions: There are **7 pages** in this test including this cover page. There should also be a spare page at the end in case you need extra room for some answers.

Ensure that **your *full* name and student number** appear on this page.

- No calculators, books, notes, or electronic devices of any kind are permitted.
- Messy work will not be graded. **Read each question carefully** to be sure you are answering the question being asked.
- Exposing your test paper, copying from another student's paper, or sharing information about this test constitutes academic dishonesty. Such behaviour may jeopardize your grade on this test, in this course, and your standing at this university.

Rules governing formal examinations:

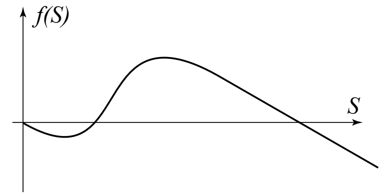
1. Every student must be prepared to produce, upon request, a UBC ID card;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Question	1	2	3	4	5	6	7	8	9	10	Total
Max	2	2	2	2	2	6	9	10	10	15	60
Actual											

For each of the following five questions, circle the best option. **If more than one is circled, no points will be given. If you change your mind, clearly indicate your final choice by writing the appropriate letter in the margin to the left.** If you need scrap paper, use the back of the page.

The first two questions relate to the dynamics of a protein that is produced by an enzyme that is up-regulated by the protein itself and might also be produced in response to an external signal (E). The equation describing the dynamics of the protein concentration S is

$$\frac{dS}{dt} = E + f(S) \qquad \text{where} \qquad f(S) = -kS + \frac{v_{max}S^2}{K_m^2 + S^2}.$$



1. (2 pts) Which of the following is a legitimate scale for S?

   (a) $\frac{v_{max}}{E}$ (b) $\frac{E}{k}$ (c) E (d) K_m^2

2. (2 pts) If the intensity of the external signal is initially zero and the protein is at a concentration $S = 0$, how large a temporary increase in E is required to increase the signal to a high level, even after the signal is removed (i.e. return to $E = 0$ but S stays high)?

   (a) $E > K_m$
   (b) $E > f(S^*)$ where S^* satisfies $f'(S^*) = 0$ and $f''(S^*) < 0$
   (c) $E > v/k$
   (d) $E > f(S^*)$ where S^* satisfies $f'(S^*) = 0$ and $f''(S^*) > 0$

3. (2 pts) Given certain conditions on a system of two differential equations, the Poincare-Bendixon Theorem allows one to determine the existence of which of the following?

   (a) A Hopf bifurcation.
   (b) An unstable steady state.
   (c) A periodic solution.
   (d) A French mathematician.

4. (2 pts) An enzyme with one binding site for a substrate S turns that substrate into a product according to the following reaction scheme:

   $$S + E \rightleftharpoons C \rightarrow P + E$$

   For such a reaction scheme, provided the total concentration of enzyme is small compared to the total concentration of substrate, the formation of product is best described by which of the following options? Assume that the parameters are appropriately consistent with definitions given in class.

   (a) $\frac{dP}{dt} = kS$ (b) $\frac{dP}{dt} = \frac{vS}{K_m + S}$ (c) $\frac{dP}{dt} = \frac{vS^2}{K_m^2 + S^2}$

   (d) $\frac{dP}{dt} = \frac{v_1S + v_2S^2}{K_{m1}K_{m2} + K_{m2}S + S^2}$

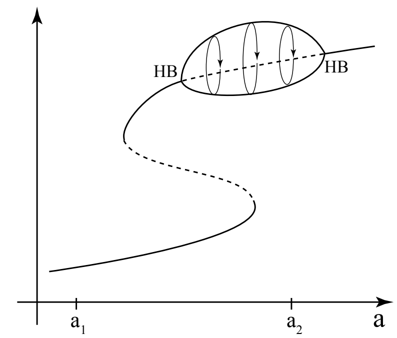
5. (2 pts) The diagram below shows the bifurcation diagram for a model of the transmembrane potential for a particular type of neuron. The parameter a represents the level of stimulation received by the neuron from neighbouring neurons and the points labeled "HB" correspond to Hopf bifurcation points. The stimulus is initially at $a = a_1$, then gradually rises to $a = a_2$ and then returns to $a = a_2$. Which of the following best describes the behaviour of the transmembrane potential of the neuron?

   (a) Steadily high, then oscillatory, then a sudden drop to a steady low level.

   (b) Steadily low followed by a transition to a steady high level and then a sudden return to a low level.

   (c) Steadily low followed by a transition to a steady high level, then oscillatory and then a sudden return to a low level.

   (d) Steadily low followed by a transition to an oscillatory mode which then settles to a steady high level followed by a sudden return to a low level.



6. (6 pts) The following **True or False** questions all refer to a population (P) whose growth dynamics show an Allee effect and is being fished at a rate proportional to the population size. The population size is described by the equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right) - EP$$

where all parameters (r, K, T, E) are positive and $T < K$. For each question, circle either **true** or **false**.

   (a) The population has three steady states for any (positive) parameter values.
   True or **False**.

   (b) The population has at least one stable steady state for any (positive) parameter values.
   True or **False**.

   (c) The population can undergo a catastrophe (sudden collapse) if the fishing effort (E) rises to a high enough level.
   True or **False**.

   (d) Whenever the population size is between T and K, the population size will be increasing.
   True or **False**.

   (e) When the population size is below T, the population size will be decreasing.
   True or **False**.

   (f) The maximum sustainable yield is attained when E is chosen so that the line $y = EP$ is tangent to the curve $y = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right)$.
   True or **False**.

7. (9 pts) Sketch the phase line for each of the following equations. Your sketch should include $\frac{dx}{dt}$ plotted against the x axis with arrows indicating direction of motion on the x axis, any stable steady states as filled circles and any unstable steady states as empty circles. For an initial condition $x(0) = \frac{1}{2}$, what eventually happens to the solution?

   (a) $\dfrac{dx}{dt} = x(1-x)$

   (b) $\dfrac{dx}{dt} = x(1-x^2)$

   (c) $\dfrac{dx}{dt} = \cos(x) + 1$

8. (a) (3 pts) **Briefly** describe a biological system that has a slow variable and a fast variable. What process is slow and what process is fast?

(b) (7 pts) Draw an approximate phase plane for the following system of equations under the assumption that $0 < \epsilon << 1$. What kind of steady state does it have?

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\epsilon}(-x+y), \\ \frac{dy}{dt} &= \frac{1}{\epsilon}(x-y)+x+y.\end{aligned}$$

9. (10 pts) Schnakenberg (1979) considered the following simplified model of glycolysis:

$$\begin{aligned}\frac{dx}{dt} &= x^2y - x\\ \frac{dy}{dt} &= a - x^2y,\end{aligned}$$

where $a > 0$. As the parameter a varies, the steady state of the system changes its behaviour. Determine the sequence of steady state classifications for increasing values of a (e.g. stable node-to-saddle-to-unstable node). Does the system undergo a Hopf bifurcation for any value(s) of a? If so, at what value(s) does the Hopf bifurcation(s) occur?

10. (14 pts) Consider the model proposed by Lotka and Volterra for the populations of two species (P_1, P_2, measured in thousands of individuals) that are in competition with each other:

$$
\begin{aligned}
\frac{dP_1}{dt} &= r_1 P_1 \left(1 - \frac{P_1}{K_1} - \frac{P_2}{\alpha_1}\right) \\
\frac{dP_2}{dt} &= r_2 P_2 \left(1 - \frac{P_2}{K_2} - \frac{P_1}{\alpha_2}\right)
\end{aligned}
$$

where $K_1 = K_2 = 1, \alpha_1 = 2$ and $\alpha_2 = 3$ (all in the same units as P_i).

(a) Consider the simple case in which the first species is the only one present ($P_1 > 0, P_2 = 0$). What eventually happens to the size of the population (P_1)?

(b) If the second species is present, what influence does it have on the growth rate of the first species? Describe the meaning of α_1.

(c) Draw the phase plane for the system. Include nullclines (label each of them clearly!), direction field arrows and steady states (as clear marked dots).

(d) If there is a stable steady state in which both species population sizes are non-zero, the species are able to coexist. In this system, can the two species coexist?