MATH 401 : Final Exam, April 13th 2017

Duration : 150 minutes.

No lecture notes, no textbooks, no calculators or electronic devices of any kind.

Content : 5 problems, Total : 50 points.

- Provide clear and justified answers.
- Write all your answers on that document.
- If you run out of space, continue on a separate sheet of paper and write your name and student number on the top. If you have many additional sheets of paper, number them. Make clear which problem they refer to. At the end of the exam, make sure you slide them in this booklet.
- Take a couple of minutes to read the whole exam first and pay attention to the tips.

First Name: Last Name:

Student-No: Section:

Signature:

Problem	1	2	3	4	5	Total
Points	8	16	6	15	5	50
Score						

Problem 1 (8pts): Green's function for ODEs

Consider the ODE boundary value problem for $u(x)$, $0 \leq x \leq \pi$, $\lambda \in \mathbb{R}$, $\lambda \leq 0$, defined as follows:

$$u'' + \lambda u = f(x) \quad , \quad 0 < x < \pi \tag{1}$$
$$u'(0) = 1 \quad , \quad u'(\pi) = 2 \tag{2}$$

1. (4pts) Find the Green's function $G_x(z)$ for this problem and give the solution formula for $u(x)$.
2. (2pts) For what values of λ does the solution break down ? In another words, for what values of λ is a condition on $f(x)$ required for there to be a solution ? Find this solvability condition.
3. (2pts) For $\lambda = 0$, formulate the problem a modified Green's function $\tilde{G}_x(z)$ solves, solve it and give the expression of $\tilde{G}_x(z)$.

Tip 1: Considering the following 1D ODE BVP: $Lu = a_0u'' + a_1u' + a_2u = f$, $x_0 < x < x_1$, *BCs at* x_0 *and* x_1, *and assuming* L *is self-adjoint, we have:*

$$(G, Lu) = (u, LG) + [a_0(Gu' - G'u)]_{x_0}^{x_1}$$

Tip 2: 2 useful trigonometric identities

$$cos(a \pm b) = cos\ a\ cos\ b \mp sin\ a\ sin\ b$$
$$sin(a \pm b) = sin\ a\ cos\ b \pm cos\ a\ sin\ b$$

Tip 3: The ODE solved by a modified Green's function contains an additional term on the rhs of the form $-\dfrac{u^*(x)}{(u^*, u^*)}u^*(z)$.

Problem 2 (16pts): Wave equation

1. (2pts) Let $D \subset \mathbb{R}^n$ be a domain bounded by a smooth surface ∂D. Find the Euler-Lagrange equation corresponding to the extremum of the following functional:

$$I(u) = \frac{1}{2}\int_0^T \int_D \left(\left(\frac{\partial u}{\partial t} \right)^2 - c^2 |\nabla u|^2 \right) d\boldsymbol{x} dt \tag{3}$$

where u belongs to the space of funtions $\mathcal{H}$ defined as:

$$\mathcal{H} = \{ u \in C^2(D \times [0,T]) \,|\, u(\boldsymbol{x},t) = g(\boldsymbol{x}) \, \text{for} \, \boldsymbol{x} \in \partial D, \, u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \, u(\boldsymbol{x},T) = u_1(\boldsymbol{x}) \} \tag{4}$$

2. (2pts) We replace the condition $u(\boldsymbol{x},T) = u_1(\boldsymbol{x})$ by the classical initial condition $\frac{\partial u}{\partial t}(\boldsymbol{x},0) = v_0(\boldsymbol{x})$. Write the corresponding problem the Green's function $G_{\boldsymbol{x},t}(\boldsymbol{y},\sigma)$ solves.

3. We now consider $n = 1$ and no boundary condition. For the ease of notation, the subscript $_{(x,t)}$ is dropped from now on.

   (a) (1pt) Write the 1D free-space problem for the Green's function.

   (b) (2pts) Using a Laplace transform in time from σ to s, assuming $G(y,0) = 0$, assuming $\frac{\partial G}{\partial \sigma} = \delta(y-x)\delta(\sigma)$ and solving for $\sigma > 0$ where $\delta(\sigma) = 0$, establish the ODE the Laplace transform of G, denoted $L_G(y,s)$, solves.

   (c) (3pts) Imposing continuity of $L_G(y,s)$ at $y = x$, using jump condition of $L_G(y,s)$ at $y = x$ and assuming $L_G(y,s)$ decays to 0 when y tends to $\pm\infty$, find the expression of $L_G(y,s)$.

   (d) (2pts) By analogy to the Laplace transform of $H(t-r)$, $r \geq 0$, where H is the Heavyside function, transform back $L_G(y,s)$ to $G(y,\sigma)$, and give the final expression of the 1D free-space Green's function for the wave equation, denoted $G^f(y,\sigma)$, and show that $G^f(y,\sigma) = \frac{1}{2c}H(\sigma - \frac{1}{c}|y-x|)$.

   (e) (1pt) Sketch a graph of $G^f(y,\sigma)$ in a (y,σ) space with an impulse at $y = x$ and comment your sketch.

4. (3pts) Finally, we consider a 1D wave equation problem in the half space $x \geq 0$. The problem read as follows:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad 0 < x < \infty \quad , \quad t > 0 \tag{5}$$

$$u(0,t) = 0 \tag{6}$$

$$u(x,0) = w(x) \; , \; \frac{\partial u}{\partial t}(x,0) = 0 \tag{7}$$

Use the method of images to solve this problem. Explain where to place the image charge and give the expression of $G_{x,t}(y,\sigma)$. Give the solution formula for $u(x,t)$. Interpret your solution in terms of travelling waves.

Tip 1: Recall that it is more convenient to formulate the Green's function problem in terms of the time variable $\sigma = t - \tau$ and then ultimately change back to τ.

Tip 2: The complete solution formula for the Dirichlet IBVP corresponding to the following wave equation in $V \subset \mathbb{R}^n$, $n = 1, 2$ or 3, smoothly bounded by ∂V:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f(\boldsymbol{x}, t) \quad &, \quad \boldsymbol{x} \in V \quad , \quad t > 0 \\ u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t) \quad &, \quad \boldsymbol{x} \in \partial V \quad , \quad t \geq 0 \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad &, \quad \frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in V \end{aligned}$$

is

$$\begin{aligned} u(\boldsymbol{x}, t) = & \int_0^t \int_V G_{\boldsymbol{x},t}(\boldsymbol{y}, \tau) f(\boldsymbol{y}, \tau) d\boldsymbol{y} d\tau + \int_V \left(G_{\boldsymbol{x},t}(\boldsymbol{y}, 0) v_0(\boldsymbol{y}) - u_0(\boldsymbol{y}) \frac{\partial G_{\boldsymbol{x},t}}{\partial \tau}(\boldsymbol{y}, 0) \right) d\boldsymbol{y} \\ & - c^2 \int_0^t \int_{\partial V} \frac{\partial G_{\boldsymbol{x},t}(\boldsymbol{y}, \tau)}{\partial n(\boldsymbol{y})} g(\boldsymbol{y}, \tau) dS(\boldsymbol{y}) d\tau \end{aligned}$$

Tip 3: The Laplace transform of an integrable function f(x), defined for all $x \geq 0$, is $L_f(s) = \int_0^{+\infty} e^{-sx} f(x) dx$ and the Laplace transform of its second derivative $f''(x)$ is given by the following formula: $L_{f''}(s) = s^2 L_f(s) - s f(0) - f'(0)$.

Problem 3 (6pts): Maximum principle for the heat equation

Let $D \subset \mathbb{R}^n$ be a domain bounded by a smooth surface ∂D.

1. (2pts) We consider the steady-state BVP for the heat equation defined as follows:

$$\Delta u = f(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in D \tag{8}$$
$$u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in \partial D \tag{9}$$

   Use the maximum principle to show that if $f(\boldsymbol{x}) \leq 0$, $g(\boldsymbol{x}) \geq 0$ (and not identically zero on ∂D), then $u(\boldsymbol{x}) > 0$.

2. (2pts) We consider the steady-state BVP for the heat equation defined as follows:

$$\Delta u = f(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in D \tag{10}$$
$$u(\boldsymbol{x}) = 0 \quad , \quad \boldsymbol{x} \in \partial D \tag{11}$$

   Use the maximum principle to show that if $f(\boldsymbol{x}) \geq 0$ then the Green's function $G_x(y)$ corresponding to this problem satisfies $G_x(y) \leq 0, \forall y \in D$.

3. (2pts) Finally we consider the IBVP for the heat equation defined as follows:

$$\frac{\partial u}{\partial t} - \Delta u = f(\boldsymbol{x}, t) \quad , \quad \boldsymbol{x} \in D \quad , \quad t > 0 \tag{12}$$
$$u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t) \quad , \quad \boldsymbol{x} \in \partial D \quad , \quad t \geq 0 \tag{13}$$
$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in V \tag{14}$$

   Use the maximum principle to show that if $f(\boldsymbol{x}, t) \geq 0$, $t \in [0, T]$, $g(\boldsymbol{x}, t) \geq 0$, $t \in [0, T]$ and $u_0(\boldsymbol{x}) \geq 0$ (and not identically zero in D), then $u(\boldsymbol{x}, t) > 0$ for $t \in [0, T]$.

Tip: In question 2, use a combination of the maximum principle applied to $u(x)$ and the solution formula of $u(x)$ involving $G_x(y)$.

Problem 4 (15pts + Bonus 2pts): Eigenvalues in a portion of a disk and bounds for other simple geometric domains

Consider the following eigenvalue problem for the operator $-\Delta$ with Dirichlet (zero) boundary conditions in a domain D of $\mathbb{R}^2$ where D is a portion of a disk of radius a:

$$\Delta u + \lambda u = 0 \text{ in } D \tag{15}$$

$$u = 0 \text{ on } \partial D \tag{16}$$

The domain D is defined in polar coordinates (r, θ) as: $0 \leq r < a$, $0 < \theta < \pi/K$ where K is an integer number. In the following, we will consider $K = 1$ (half-disk) and $K = 2$ (quarter of a disk). The objective is to solve this problem with Bessel functions and then to use the results to bound eigenvalues for other simple domains by bounding the domain or using simple test functions.

1. We start by setting $K = 1$ and hence we pose the problem in the half-disk illustrated in Figure 1.

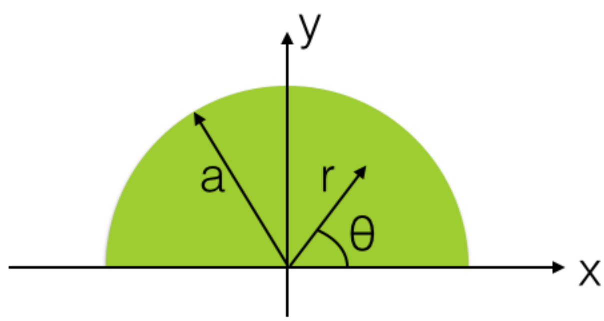

Figure 1 A half disk domain

   (i) (1pt) Formulate the eigenvalue problem in polar coordinates with corresponding boundary conditions.

   (ii) (3pts) Solve that problem by seeking a solution of the separate variables form $\phi(r, \theta) = F(r) sin(n\pi\theta/\alpha)$, $n = 1, 2, \ldots$ with $\alpha = \pi/K$, $K = 1$. Express the solution in terms of Bessel functions and give the expression of the eigenvalues as a function of the zeros of the appropriate Bessel functions.

   (iii) (1pt) Using the fact that the first zero of $J_1(z)$ is $z_{1,0} = 3.83$, give the expression for the first (lowest) eigenvalue.

2. (2pts) We now consider a domain D corresponding to a half-ellipse in the positive half-space $y \geq 0$, i.e., ∂D is defined by $y = 0$, $x \in [-a, a]$, and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y > 0$, $x \in [-a, a]$, with a and b 2 positive non-zero numbers and $b > a$. Sketch the domain D and bound the first (lowest) eigenvalue of D by bounding the geometry with appropriate half-disks.

3. (2pts) We now consider a domain D corresponding to the quarter of a disk, i.e., $K = 2$, illustrated in Figure 2. Explain what changes in the solution with respect to Question 1 and using the fact that the first zero of $J_2(z)$ is $z_{2,0} = 5.14$, give the expression for the first (lowest) eigenvalue.

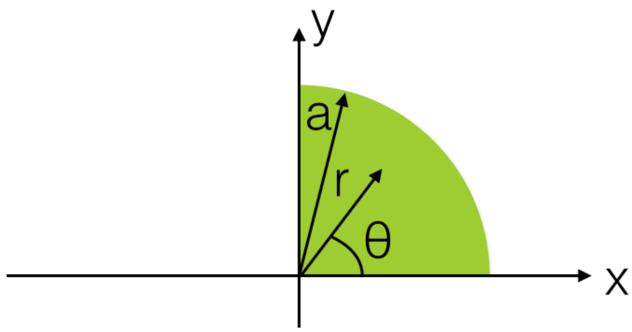

Figure 2 A quarter of a disk domain

4. We now consider a domain D corresponding to the triangle illustrated in Figure 3 and defined by the corners $(0,0)$, $(1,0)$ and $(0,1)$.

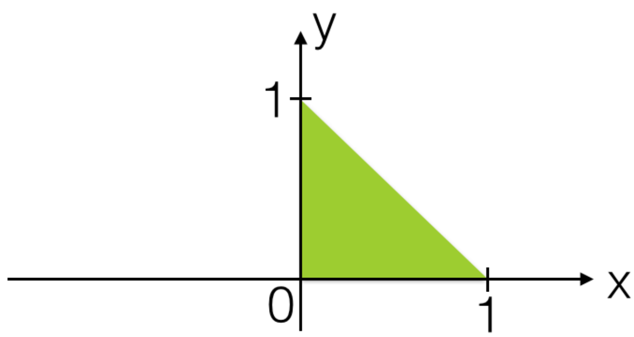

Figure 3 The triangular domain

(i) (1pt) Find upper and lower bounds for the first (lowest) eigenvalue by bounding the geometry with appropriate disks. Recall that the lowest eigenvalue of the operator $-\Delta$ with Dirichlet (zero) boundary conditions on a disk of radius a is approximately given by $5.76/a^2$.

(ii) (1pt) Find upper and lower bounds for the first (lowest) eigenvalue by bounding the geometry with appropriate quarters of a disk.

(iii) (1pt) Find upper and lower bounds for the first (lowest) eigenvalue by bounding the geometry with appropriate squares.

(iv) (Bonus +2pts) Although the choice of the largest square inside the triangle is obvious, it is less obvious that a square would give the lowest upper bound for the first (lowest) eigenvalue. Write the upper bound for the first (lowest) eigenvalue using a rectangle inscribed in the triangle with its upper right corner located on the largest side of the triangle defined by the equation $x+y=1$. Formulate the problem of finding this lowest upper bound as a constrained minimization problem. Solve it (without Lagrange multiplier) and show that the lowest upper bound indeed corresponds to a square.

(v) (2pts) Suggest a simple admissible test function constructed as a product of 3 linear functions in x and y to provide another upper bound for the first (lowest) eigenvalue using the Rayleigh quotient. Write the expression of the Rayleigh quotient for this test function but do not carry out the integration.

(vi) (1pt) Summarize the different bounds you obtained for the first (lowest) eigenvalue for the triangle and give the best ones.

Tip 1: The inscribed disk in a triangle (largest disk contained in a triangle) has a radius of $2S/P$ *where* S *is the surface area of the triangle and* P *its perimeter.*

Tip 2: $(2+\sqrt{2})^2 \simeq 11.65$, $3.83^2 \simeq 14.67$, $5.14^2 \simeq 26.42$, $\pi^2 \simeq 9.87$

Tip 3: The solution of $F''(r) + \frac{1}{r}F'(r) + (\lambda - \frac{n^2}{r^2})F(r) = 0$ *can be written in terms of Bessel functions as* $F(r) = AJ_n(\pm\sqrt{\lambda}r) + BY_n(\pm\sqrt{\lambda}r)$ *with* A *and* B *two constants. Also recall that* $J_n(x) = J_n(-x)$, *and* $Y_n(x) \to \infty$ *as* $x \to 0$.

Tip 4: In question 4(v), it is recommended to rewrite $|\nabla u|^2 = \nabla u \cdot \nabla u$ *in terms of* u *and* Δu, u *being the test function.*

Tip 5: the solution to the equation $-1 + 3x - 3x^2 + 2x^3 = 0$ *for* $x \in [0,1]$ *is* $x = 0.5$.

Problem 5 (5pts): Shortest curve between two points in 2D

Consider the problem of determining the shortest curve between point $A(x_A, y_A)$ and point $B(x_B, y_B)$.

1. (2pts) Sketch the problem and formulate it as a minimization problem (functional to minimize and boundary conditions).
2. (2pts) Determine the Euler-Lagrange (EL) equation for that problem.
3. (1pt) Solve the problem (EL equation + BCs) and show that the minimizer is a straight line (a result that, I hope, you already knew !).