Be sure this exam has 8 pages including the cover

The University of British Columbia

Sessional Exams – 2010 Winter Term 1
Mathematics 418 Probability

Name: ______________________________

Student Number: ______________________________

This exam consists of **7** questions worth **10** marks each. No aids other than nonprogrammabble calculators are permitted. Answers must be accompanied by proofs/explanations unless the question says otherwise.

Problem	score
1.	
2.	
3.	
4.	
5.	
6.	
7.	
total	

1. Let X, Y be random variables on $(\Omega, \mathcal{F}, P)$.

   (a) Define "random variable".

   (b) Prove that $Z = |X|$ is a r.v.

   (c) Let N be an integer, let $\Omega = \mathbb{R}$ and let $\mathcal{F}_N$ be the smallest σ-field that contains all the intervals $(\frac{i-1}{N}, \frac{i}{N}]$, where $i \in \mathbb{Z}$. Describe, without proof, the random variables in this case.

2. There are n couples, each a husband and wife, at a dance. The name of each husband is written on a card, the cards are then shuffled and dealt to the wives. Each wife then dances with the man whose card she holds. Let E_i be the event that wife i dances with her own husband and let p_n be the probability that no married couple dances together.

   (a) Find p_2 by expressing it in terms of the events E_i.

   (b) Find p_3.

   (c) Is $\lim_{n\to\infty} p_n = 0$?

3. Let Y be a continuous random variable with density $f_Y(y) = \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{else.} \end{cases}$

   (a) What is the distribution of Y

   (b) Let X have the uniform distribution on $[0, 1]$. Find a function g such that $g(X)$ equals Y in distribution. Does g have to be defined on all of $\mathbb{R}$?

4. Suppose that the number X of cars that cross the border into the USA in a given hour is a Poisson random variable with parameter $\lambda > 0$. Independently, each of these cars has probability p of being searched. Let Y be the number of cars that are searched in the specified hour.

   (a) What is $E(Y|X)$?

   (b) Find $P(Y = k)$.

(10 points) 5. George and Julia work at the campus coffee shop. The management wants to award a prize to the quicker worker. They will each be set the task of making 200 consecutive double decaf skim milk lattes and, for each, the total of the 200 independent times to accomplish this will be measured. If the two total times differ by more than 80 seconds, the prize will be awarded to the faster one, otherwise no prize will be awarded. The standard deviation of the time it takes each person to make a double decaf skim milk latte is 4 seconds. If the mean times for both George and Julia are actually the same, what is the approximate probability that George will get the prize? You can give your answer as a simple formula.

6. Let $E_n(b)$ be the event that random walk hits $b \in \mathbb{Z}$ for the first time at step n. The hitting time theorem for symmetric random walk says that $P(E_n(b)) = \frac{|b|}{n} P(S_n = b)$, where the walk starts from the origin and $n \geq 1$. Let $T = \{n \geq 1 : S_n = 0\}$ be the time of first return to the starting point. Find $P(T = 2n)$ in a simple form.

(10 points) 7. Alice and Bob are playing a series of games numbered $k = 1, 2, \ldots$. In the kth game, Alice either wins \$$k$ from Bob or loses \$$k$ to B, both with probability $1/2$. Let S_n be the total amount of money that Alice has won from Bob in the first n games including game n. If she has lost money then S_n is negative.

(a) What does it mean to say that a sequence Z_n of random variables converges in distribution to a random variable Z?

(b) If Z_n converges in distribution to Z, when does it also converge in probability? (No proof).

(c) Let $Z_n = n^{-p} S_n$. Find p such that the variance of Z_n has a non-zero finite limit as $n \to \infty$.

(d) Prove that the sequence Z_n converges in distribution to a random variable Z and find the distribution of Z. Limits of sums can sometimes be evaluated by recognising them to be Riemann sums.