Be sure that this examination has 2 pages.

The University of British Columbia

Sessional Examinations - April 2006

Mathematics 419

Stochastic Processes

Closed book examination — Time: $2\frac{1}{2}$ hours

Special Instructions: No calculators, books or notes are allowed. $\mathbf{R}$ is the set of real numbers, $\mathbf{Z}$ is the set of integers and $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$.

Marks

[10] **1.** State carefully:

(a) The Strong Law of Large Numbers.

(b) The Kolmogorov Zero-one Law.

[5] **2.** Give an example of a martingale $\{M_n\}$ which converges $a.s.$ as $n \to \infty$ but does not converge in L^1. *Briefly* justify your example.

[10] **3.** True or False. If True, give a proof; if False, give a counter-example.

(a) If T is an $(\mathcal{F}_n)$-stopping time, then so is $2T$.

(b) If $X_n \stackrel{L^p}{\to} X$ for some $p > 1$, then $X_n \stackrel{L^1}{\to} X$.

[15] **4.** Let $X = \{X_n : n \in \mathbf{Z}_+\}$ be a Galton-Watson branching process with offspring distribution $P(N = n) = p_n,\ n = 0, 1, 2, \ldots$ satisfying $p_1 < 1$ and $E(N^2) < \infty$. Recall that X is a $\mathbf{Z}_+$-valued Markov chain.

(a) Classify each state in $\mathbf{Z}_+$ as transient, null recurrent or positive recurrent. Justify your answers.

(b) If $p_n = p^n(1-p)$ for $n = 0, 1, \ldots$, for some $0 < p < 1$, and $X_0 = 1$, find $P(X$ becomes extinct$)$.

[8] **5.** Show that any L^2-bounded $(\mathcal{F}_n)$-martingale, M_n, may be written as the difference of two non-negative L^2-bounded $(\mathcal{F}_n)$-martingales.

[5] **6.** Assume $X = \{X_t : t \geq 0\}$ is a Lévy process such that X_t is integrable and has mean 0. Prove X is a martingale.

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[8] **7.** Let X_1, X_2 be independent identically distributed r.v.'s, each taking on the values 0 and 1 with probability 1/2. Let $Y = X_1X_2$.

(a) Find $E(X_1|Y)$.

(b) Show that in general $E(X_1X_2|\mathcal{G})$ may not be a.s. equal to $E(X_1|\mathcal{G})E(X_2|\mathcal{G})$ for a sub-σ-field $\mathcal{G}$.

[14] **8.** (a) State the Borel-Cantelli Lemma.

(b) Assume $\{X_n : n \geq 1\}$ are i.i.d. Cauchy r.v.'s. Recall that a Cauchy r.v. has density $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbf{R}$. Prove that $\limsup_{n\to\infty} \frac{\log|X_n|}{\log n} = 1$ a.s.

[11] **9.** Consider a connected, locally finite graph with vertex set S countably infinite. By connected we mean that for any $i, j \in S$ there is a finite set of edges $i = i_0, i_1, i_2, \ldots, i_n = j$, such that there is an edge from i_k to i_{k+1} for $k = 0, \ldots, n-1$. Locally finite means for each vertex i, the number of vertices connected to i by a single edge is finite. Assume also no vertex is connected to itself. Let $X = \{X_n : n \in \mathbf{Z}_+\}$ be a simple random walk on the graph.

(a) Find a stationary measure for X. Justify your answer.

(b) Prove that either all states are transient, or all states are null recurrent.

(c) Give examples of each possibility in (b). You need not justify your examples.

[22] **10.** Let $\{X_n : n \in \mathbf{Z}_+\}$ be an aperiodic irreducible Markov chain with finite state space S. [Remember to use any results from the course in this question.]

(a) Show there is a natural number n_0 such that $\min_{i,k\in S} p_{i,k}(n_0) = \rho > 0$.

(b) If $T_k = \min\{n \geq 1 : X_n = k\}$, prove that for all $i, k \in S$, and all natural numbers n, $P_i(T_k > n_0 n) \leq (1-\rho)^n$.

(c) Prove that X has a stationary distribution π.

(d) Prove that there is a $\lambda > 0$ and $c > 0$ so that for all $i \in S$, and all natural numbers n,

$$\sum_{j\in S} |\pi_j - P_i(X_n = j)| \leq ce^{-\lambda n}.$$

[108] **Total Marks**

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