Mathematics 420 / 507 — Real Analysis / Measure Theory

Final Exam — Wednesday 14 December 2005, 8:30 am (2 hours 30 minutes)

All 5 questions carry equal credit. No calculators, books or notes allowed.

(1) (a) For a measure space $(X, \mathcal{M}, \mu)$ and $p \in [1, \infty)$ define: (i) $\|f\|_p$; (ii) L^p; (iii) convergence in L^p.

(b) State the Hölder and Minkowski inequalities. (You do not need to say when equality holds).

(c) Let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Show that if $f, f_1, f_2, \ldots \in L^p$ satisfy $f_n \to f$ in L^p and $g, g_1, g_2, \ldots \in L^q$ satisfy $g_n \to g$ in L^q, then $f_n g_n \to fg$ in L^1. (Here fg denotes the pointwise product).

(d) For some $p, q \in (1, \infty)$ with $1/p + 1/q \neq 1$ give an example to show that the implication in (c) need not hold.

(2) Let μ, ν, λ be σ-finite *positive* measures on $(X, \mathcal{M})$.

(a) Show that $\mu \ll \mu + \nu$.

(b) Show that if $\nu \ll \mu$ and $\lambda \ll \mu$ then $\nu + \lambda \ll \mu$ and

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \quad \mu\text{-a.e.}$$

(c) Show that if $\lambda \ll \nu \ll \mu$ then $\lambda \ll \mu$ and

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \frac{d\nu}{d\mu} \quad \mu\text{-a.e.}$$

(d) Show that if $\lambda \ll \mu$ and $\lambda \ll \nu$ then $\lambda \ll \mu + \nu$; find and prove a formula for $\dfrac{d\lambda}{d(\mu + \nu)}$ in terms of (only) $\dfrac{d\lambda}{d\mu}$ and $\dfrac{d\lambda}{d\nu}$, assuming that $\dfrac{d\lambda}{d\mu}, \dfrac{d\lambda}{d\nu} \in (0, \infty)$.

(3) (a) State: (i) the monotone convergence theorem; (ii) Fatou's lemma; (iii) the dominated convergence theorem.

(b) Assuming (ii), prove (iii).

(c) Evaluate $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{x + x^2}\, dx$, justifying your answer.

(4) Let m denote Lebesgue measure on $\mathbb{R}^2$.

(a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable then

$$m\{(x, f(x)) : x \in \mathbb{R}\} = 0.$$

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable then

$$m\{(x + f(x), x - f(x)) : x \in \mathbb{R}\} = 0.$$

Hint: apply a transformation of $\mathbb{R}^2$.

(c) Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are *increasing* then

$$m\{(f(t), g(t)) : t \in \mathbb{R}\} = 0.$$

Hint: consider the intersection of the set with the line $\{(x, y) : x + y = a\}$.

(5) Let $f, f_1, f_2, \ldots$ be measurable real functions on $(X, \mathcal{M}, \mu)$. For $A \subset X$, recall that "$f_n \to f$ uniformly on A" means that for every $\epsilon > 0$ there exists N such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq N \text{ and } x \in A.$$

We say that "$f_n \to f$ *almost uniformly*" if for every $\delta > 0$ there exists $A \in \mathcal{M}$ with $\mu(A^C) < \delta$ such that $f_n \to f$ uniformly on A.

(a) Show that if $f_n \to f$ almost uniformly then $f_n \to f$ almost everywhere.

(b) Suppose $\mu(X) < \infty$. Show that if $f_n \to f$ almost everywhere then $f_n \to f$ almost uniformly.

(Hints: Let $E(\epsilon, N)$ be the set of x such that $|f_n(x) - f(x)| > \epsilon$ for some $n \geq N$. Show that $\lim_{N\to\infty} \mu(E(\epsilon, N)) = 0$. Then choose N_k such that $\mu(E(1/k, N_k)) \leq \delta 2^{-k}$.)

(c) Give an example to show that if $\mu(X) = \infty$ then the implication in (b) need not hold.