*This final exam has **9 questions** on **10 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________

Student-No: ______________________

Signature: ______________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	12	12	12	8	8	10	10	13	100
Score:										

Please read the following points carefully before starting to write.

- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

15 marks 1. Short answers:

(a) Define carefully: a nowhere dense subset of a topological space X.

(b) State carefully: the Baire category theorem.

(c) Show that a metric space is first countable.

12 marks 2. Give examples of the following. *You need not justify your examples.*

(a) A metric on $X = \mathbb{R}^{\mathbb{N}}$ which induces the product topology on X.

(b) A function f in L^2 such that $\Phi_f(g) = \int f(x)g(x)dm$ does not define a bounded linear functional on $L^{3/2}$. Here m is Lebesgue measure on $[0,1]$ and $L^p = L^p(m)$.

(c) An unbounded closed linear operator from one normed linear space to another. (Carefully specifying the normed linear spaces is part of the question.)

12 marks 3. Let A and B be bounded linear operators on a Hilbert space $\mathcal{H}$.

(a) If A is self-adjoint and λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenspaces $E(\lambda_1)$ and $E(\lambda_2)$, then prove these eigenspaces are orthogonal.

(b) If A is compact, prove that AB and BA are both compact.

12 marks 4. (a) Carefully state the Riesz Representation Theorem for positive linear functionals on the space of continuous functions on a compact Hausdorff space, X.

(b) Let X be a compact Hausdorff space. By a probability on X we mean a Borel measure on X so that $\mu(X) = 1$. Assume $\{\mu_n\}$ is a sequence of probabilities on X such that $\lim_n \int f d\mu_n = I(f)$ exists for every $f \in C(X)$. Prove that there is a regular probability on X so that $lim_n \int f d\mu_n = \int f d\mu$ for every $f \in C(X)$.

$\boxed{\text{8 marks}}$ 5. Assume X is a Hausdorff topological space and $\{x_\alpha : \alpha \in A\}$ is a convergent net in X. Prove that $\lim_\alpha x_\alpha$ is unique.

$\boxed{\text{8 marks}}$ 6. True or False. If true, give a proof. If false, provide a counter-example. You may use any results proved in the lectures.
If $\mathcal{X}$ is a Banach space, then any sequence in $\mathcal{X}^*$ which converges in the weak-$*$ topology is bounded.

$\boxed{\text{10 marks}}$ 7. (a) Carefully state the spectral theorem for compact self-adjoint operators on a Hilbert space, $\mathcal{H}$.

(b) Let A be a compact self-adjoint operator on a non-separable Hilbert space $\mathcal{H}$. Prove that 0 is an eigenvalue of A.

10 marks 8. (a) Carefully state the Open Mapping Theorem.

(b) Let $\mathcal{X}$ be a Banach space in either of the two norms $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that $\|x\|_1 \le C\|x\|_2$ for all $x \in \mathcal{X}$, for some positive constant C. Prove that there is a positive constant D so that $\|x\|_2 \le D\|x\|_1$ for all $x \in \mathcal{X}$.

13 marks 9. (a) State the real vector space version of the Hahn-Banach Theorem.

(b) Let ℓ_∞ denote the real vector space of bounded real-valued sequences equipped with the usual norm $\|\{a_n\}\|_\infty = \sup\{|a_n| : n \in \mathbb{N}\}$. Recall that for $\{a_n\} \in \ell_\infty$, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\}$ and similarly for $\liminf_{n\to\infty} a_n$.

Prove there is a bounded linear functional f on ℓ_∞ so that $\|f\| = 1$ and

$$\liminf_{n\to\infty} a_n \leq f(\{a_n\}) \leq \limsup_{n\to\infty} a_n \text{ for all } \{a_n\} \in \ell_\infty.$$

[MORE SPACE ON NEXT PAGE]

SPACE FOR MORE WORK