# Final Exam

**For simplicity, all rings are commutative with unit.**

**Problem 1.**
Let $\mathcal{A}$ be an abelian category.

(a) Explain what a chain complex in $\mathcal{A}$ is.
(b) Explain what the homology of a chain complex is.
(c) Explain what a homomorphism of chain complexes is.
(d) Explain what a chain homotopy is.
(e) Prove that chain homotopic homomorphisms induce indentical homomorphisms on homology.

**Problem 2.**
Let $R$ be a ring and $M$, $N$ two $R$-modules. Explain how the $R$-modules $\operatorname{Ext}^i_R(M, N)$ are constructed.

**Problem 3.**
Let $R$ be a ring.

(a) Explain what a non zero divisor in $R$ is.
(b) Define the term *projective dimension* of an $R$-module $M$.
(c) Suppose $x$ is a non zero divisor in $R$. Prove that $R/xR$ has projective dimension 1.
(d) Give an example of a ring $R$ and a module $M$, such that the projectiv dimension of $M$ is infinite.

**Problem 4.**
Consider a ring $R$.

(a) Define the term *global dimension* of $R$.
(b) Explain why the global dimension of $\mathbb{Z}$ is 1.
(c) Give an example of a ring with infinitie global dimension.

**Problem 5.**
Suppose that $f : X \to Y$ is a 'fibration' of topological spaces, with fibre $F$. Suppose further, that sufficient hypotheses are satisfied, such that the Leray spectral sequence of $f$ reads

$$E_2^{p,q} = H^p(Y, \mathbb{Q}) \otimes H^q(F, \mathbb{Q}) \Longrightarrow H^{p+q}(X, \mathbb{Q})$$

(a) Suppose that $H^i(Y, \mathbb{Q}) = \mathbb{Q}$, for $i = 0, 2, 4$, and 0 otherwise. Suppose that $H^i(F, \mathbb{Q}) = \mathbb{Q}$, for $i = 0, 3$, and 0 otherwise. Display graphically the $E_2$-term of this Leray spectral sequence in this case.
(b) What can you conclude about the cohomology of $X$, under these assumptions?