The University of British Columbia

Math 437/537 — Number Theory

Fall 2011

Instructor: Dragos Ghioca

First Name: ______________________ **Last Name:** ______________________

Student id. ____

Instructions

• This exam consists of **7 questions** worth a total of 50 points. Make sure your exam has all the questions.
• Justify all answers, show all work and calculations. If you need additional space for writing your solutions, use the last page of your exam.
• No calculators or other aids are permitted.
• Duration: 150 minutes.

1. Each candidate should be prepared to produce his library/UBC card upon request.
2. Read and observe the following rules:
CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
(b) Speaking or communicating with other candidates.
(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

Question:	1	2	3	4	5	6	7	Total
Points:	5	7	5	10	7	6	10	50
Score:								

(5 marks) 1. Let n be an integer such that $n > 1$. Prove that $n \nmid (2^n - 1)$.

(7 marks) 2. Let $k \in \mathbb{N}$, let $a_0, a_1, \ldots, a_k \in \mathbb{R}$ with $a_k \neq 0$. If $f : \mathbb{N} \longrightarrow \mathbb{R}$ given by

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

is a multiplicative function, show that $a_k = 1$, and that $a_i = 0$ for $0 \leq i < k$.

(5 marks) 3. Show that there exist arbitrarily long sequences of consecutive positive integers such that no integer in the sequence is a power of a prime number.

(10 marks) 4. Let $p \geq 5$ be a prime number. Let $a, b \in \mathbb{N}$ such that

(i) $\gcd(a, b) = 1$; and

(ii)

$$\sum_{i=1}^{p-1} \frac{1}{i} = \frac{a}{b}.$$

Prove that $p^2 \mid a$.

(7 marks) 5. Let $m, n \in \mathbb{N}$ such that $m \cdot \phi(m) = n \cdot \phi(n)$. Prove that $m = n$.

(6 marks) 6. If $n \in \mathbb{N}$ is an even perfect number, then there exists $k \in \mathbb{N}$ such that $\left(2^{k+1} - 1\right)$ is a prime number, and

$$n = 2^k \cdot \left(2^{k+1} - 1\right).$$

(10 marks) 7. Let $a \in \mathbb{Z}$ such that $a \equiv 5 \pmod 8$, and let $\alpha \in \mathbb{N}$.

(i) Prove that the order of a modulo $2^{\alpha+2}$ is 2^{α}.

(ii) For each odd integer x prove that there exist a unique $i \in \{0, 1\}$ and a unique $j \in \{0, 1, 2, 3, \ldots, 2^{\alpha} - 1\}$ such that

$$x \equiv (-1)^i a^j \pmod{2^{\alpha+2}}.$$

Question: ________