math 516: Final exam

december 3 2008

1 Neumann/Dirichlet boundary value problem

We consider the following set of $\mathbb{R}^2$:

$$U := (-1,1) \times (-1,1) := \{(x,y) \in \mathbb{R}^2 \text{ such that } -1 < x < 1, -1 < y < 1\},$$

and $\partial U = \Gamma_1 \cup \Gamma_2$ with

$$\Gamma_1 := (\{-1\} \cup \{1\}) \times (-1,1) = \{(x,y) \in \mathbb{R}^2 \text{ such that } |x| = 1, -1 < y < 1\}.$$

$$\Gamma_2 := (-1,1) \times (\{-1\} \cup \{1\}) = \{(x,y) \in \mathbb{R}^2 \text{ such that } |y| = 1, -1 < x < 1\}.$$

and we want to solve on U the following equation, with $f \in L^2(U)$:

$$-\Delta u = f \; in \; U \tag{1}$$
$$u = 0 \; on \; \Gamma_1 \tag{2}$$
$$\frac{\partial u}{\partial n} = 0 \; on \; \Gamma_2 \tag{3}$$

We define the following functional spaces:

$$E := \{u \in C^\infty(\bar{U}) \text{ such that } supp(u) \cap \Gamma_1 = \emptyset\}$$

$$H := \{u \in H^1(U) \text{ such that } u|_{\Gamma_1} = 0\}.$$

1. Show that the closure of E in $H^1(U)$ is H.
2. Prove the following Poincaré inequality

$$\|u\|_{L^2(U)} \leq \sqrt{2}\|\nabla u\|_{L^2(U)}, \forall u \in H.$$

   Hint: Prove it first for $u \in E$, by writing $u(x,y) = \int_{-1}^{x} \frac{\partial u}{\partial x}(s,y)ds$.

3. Show that H, endowed with the following inner product is a Hilbert space.

$$< u, v >_H := \int_U \nabla u \cdot \nabla v.$$

4. $u \in H$ is said to be a weak solution of (1) with the boundary conditions (2),(3) if we have

$$\int_U \nabla u \cdot \nabla v = \int_U fv, \forall v \in H. \tag{4}$$

   Prove that if $u \in C^2(\bar{U})$ verifies (4), then u is a strong solution of (1) with the boundary conditions (2),(3).

5. Show that $\forall f \in L^2(U)$ there exists a unique weak solution of (4).

6. Show that $T : L^2(U) \to L^2(U)$ such that $T(f)$ is a the unique weak solution of (4), is a linear compact application.

2 Nonlinear problem

We consider here the same PDE, except that this time f depends on u:

$$\begin{aligned} -\Delta u &= f(u) \; in \; U \\ u &= 0 \; on \; \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \; on \; \Gamma_2, \end{aligned} \tag{5}$$

with $f \in C^\infty(\mathbb{R})$ and lipschitz ($\|f'\|_{L^\infty(\mathbb{R})} < +\infty$).
u is a weak solution of (5) with boundary conditions (2),(3) if

$$\int_U \nabla u \cdot \nabla v = \int_U f(u)v, \forall v \in H. \tag{6}$$

In the whole problem, we consider that f verifies the following condition:

$$\|f'\|_{L^\infty(\mathbb{R})} < \frac{1}{2}. \tag{7}$$

1. Show that u is a weak solution of (6) if and only if u is a critical point of $I : H \to \mathbb{R}$ defined by

$$I(u) = \int_U \frac{|\nabla u|^2}{2} - F(u),$$

   with $F' = f$ and $F(0) = 0$.

2. Prove that I is coercive and bounded below.(You can use the Poincaré inequality proved in the first problem).

3. Show that I is weakly lower semi continuous on H.

4. Prove that there exists at least one weak solution of (6).

5. Show that there exists a constant $C > 0$ (depending on $\|f'\|_{L^\infty(\mathbb{R})}$) such that for any weak solution of (6)

$$\|u\|_H \leq C|f(0)|.$$

Hint: set $u = v$ in (6).

6. Prove that if $V \subset\subset U$ ($\bar{V}$ is a compact subset of U), then a weak solution u of (6) verifies the following

$$\|u\|_{H^2(V)} \leq C_V|f(0)|.$$

where C_V depends only on V and $\|f'\|_{L^\infty(\mathbb{R})}$.

7. Show that if $f(0) = 0$, then the only weak solution (not necessarily minimizer) of (6) is zero.

8. According to question 5, what lower bound can you give for the first egenvalue of $-\Delta$ on U, with the boundary conditions (2),(3)?

9. Using the function

$$(x, y) \longmapsto cos(\frac{\pi}{2}x)$$

give an upper bound for the first eigenvalue of $-\Delta$.