MATH 545, take home exam Due 2014-04-27, 23:59

Please let me know by email if there is any error or ambiguity.

Problem 1. Let a_n be a sequence of positive numbers. We construct an infinite graph as follows. We start with vertex 0. Vertex n is connected to a single random previous vertex, and is connected to vertex $i < n$ with probability proportional a_i.

Let H_n be the distance in the graph from n to 0. Show that H_n converges in distribution if and only if $\sum a_n < \infty$, and otherwise $H_n \to \infty$ in distribution.

Problem 2. Consider a Poisson process in $\mathbb{R}^2$ and let L_n be the maximal length of an increasing set of points in $[0, n]^2$ (i.e. points (x_i, y_i) of the process with x_i and y_i both increasing.
a. Prove that L_n/n converges a.s. to a constant α.
b. Show that the number of points in the square is with high probability in $[(1-\varepsilon)n^2, (1+\varepsilon)n^2]$.
c. Combine these to show that the maximal increasing subsequene in a random permutation on n elements is w.h.p. close to $\alpha\sqrt{n}$

Problem 3. Consider a continuous time Markov chain X_t on $\mathbb{N}$ with jump rates $q_{n,n+1} = \lambda$ and $q_{n,n-1} = \mu$. When is this recurrent? Find the stationary distribution when it exists. Repeat this for $q_{n,n-1} = n$.

Problem 4. a. For a Poisson process N_t with intensity λ, find For which a is $N_t - at$ a martingale? sub-martingale? super-martingale?
b. Is there a function $f(t)$ so that $e^{N_t - f(t)}$ is a martingale?

Problem 5. Let W be a standard Brownian motion with $W(0) = 0$. Let $0 < s < t$. a. What is the distribution of $W(s) + W(t)$?
b. $X_t = W(t) - tW(1)$ is called the Brownian bridge (for $t \in [0, 1]$). Find the covariance of X_t and X_s.
c. Show that the distribution of $W(s)$ conditioned on $W(t) = b$ is $N\left(\frac{s}{t}b, \frac{s}{t}(t-s)\right)$.

Problem 6. Let $\{T_n\}$ be the time at which Brownian motion first has the value n.
a. Show that T_n has the same distribution as $n^2 T_1$.
b. Show that T_n is a Markov chain, and that $T_{n+1} - T_n$ are i.i.d. for integers $n > 0$.
c. What is the limit of T_n/n as $n \to \infty$?
d. What does the law of large numbers say about T_n?

Problem 7. Let (S_i) be a simple random walk of length n on the integers, and let M_n be the maximal length of a sub-sequence $i_0 < i_1 \dots$ so that S_{i_j} (weakly) inreasing.
a. Prove that $\mathbb{E}M_n = o(n)$.
b. If we see the values of S one at a time, and deide for each whether we take it for our sub-sequence or not without looking at later terms in the sequence, and cannot go back, show that there is no strategy that does better than $C\sqrt{n}$. (Hint: what is the probability it takes therandom walk time at least t to return to its starting point?)
c*. Prove that $\mathbb{E}M_n \leq Cn^{3/4+\varepsilon}$ for any $\varepsilon > 0$, with some C depending only on ε.
d**. Improve the bound of c.